CHAPTER

3

Models and Important Results

Preview

In Chapter 2 z-transforms were introduced, and we observed a major benefit of their use—the transformation of linear difference equations that represent discrete-time systems into linear algebraic equations. Thus, we can characterize linear discrete-time systems in either the time or z-domains. As a result, there are various forms of mathematical models that can be used to represent linear discrete-time systems. In this chapter, we consider several of these models. Some of these models are expressed in the time domain and others in the z-domain. It is important to be able to take a model in one form and convert it to another, because in analyzing and designing systems, a particular model form may lead most directly to a desired result. This is an idea we will introduce in this chapter, and it will be continued and extended in following chapters. Throughout this chapter, with the sole exception of Section 3.6.6, we will be dealing with causal systems.

We begin with linear difference equations, a time-domain model, and show how the use of z-transforms enables us to find another time-domain model, the unit impulse response. The relationship between unit impulse response and the transfer function model then is revealed with the z-transform again playing a central role. From Chapter 2, we know that one way to determine a system's output is to evaluate the convolution sum, a time-domain operation. Here we observe that an alternative path is to do the heavy lifting of convolution as algebra in the z-domain, followed by an inverse z-transform back to the time domain. We also introduce system diagrams that reveal the structure of systems and often are used in representing the necessary interconnections of standard building blocks for system implementation. The algorithm Mason's Gain Rule is considered as a means for determining transfer functions from a given system diagram.

The final model form introduced in this chapter is the state-space model—a time-domain representation that is very useful for both computer solutions and theoretical investigations. The state-space approach to analysis and design of discrete-time linear systems has been applied extensively in control theory and in the description of digital filters. Its leading advocate was Rudolf E. Kalman, who, as a graduate student

Figure 3-1: Lotfi A. Zadeh

at Columbia University under the leadership of Professor John R. Ragazzini in the late 1950s, demonstrated the need for an alternative to traditional frequency-domain methods that had their roots in communication systems using the results of Bode, Nyquist, and others. The application of state-variable methods broadened with the appearance of the landmark work, *Linear System Theory*: *The State Space Approach*, by Lotfi A. Zadeh (Figure 3-1) and Charles A. Desoer in 1963, published by McGraw Hill Book Company. Today, state-variable methods provide an important tool for system design.

An underlying property that is required in virtually all systems is *stability*. We consider how to test systems represented by various models to ensure that they are stable, and demonstrate the important relationship between the region of convergence and stability. Consideration of stability also leads to recognition that an important system attribute is known by several names: characteristic roots, poles, and system eigenvalues.

3.1 The Linear Difference Equation

From Chapter 2, we recall that two concise representations of the N^{th}-order difference equations (DEs) are[1]

Recursive DEs of order N

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k)$$

and

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k).$$

These N^{th}-order difference equations represent an important class of discrete-time systems known as *recursive*, because the output $y(n)$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{L} b_k x(n-k)$$

depends on previous values of the output as in $a_k y(n-k)$, $k = 1, \ldots, N$ as well as on the current and past values of the input as in $b_k x(n-k)$, $k = 0, 1, \ldots, L$.

Another class of discrete-time systems is described by

$$y(n) = \sum_{k=0}^{L} b_k x(n-k)$$

and is known as *nonrecursive* because the previous values of the output do not enter the calculations. Figure 3-2a gives a pictorial version of a specific second-order recursive system

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1)$$

and Figure 3-2b shows the representation of a nonrecursive system having three delays of the input $x(n)$ giving the output $y(n)$ as

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3).$$

In the system diagrams of Figure 3-2a, box enclosing a D indicates a unit delay, a triangle represents a multiplier whose coefficient is the number contained, and a plus sign in a circle represents a summing operation. A thorough discussion of system diagrams may be found in Section 3.4.

3.2 The Unit Impulse Response Model

Difference equations are one way of modeling linear, time-invariant, discrete-time systems. A second model is the unit impulse response, the zero-state output produced by applying an input of $x(n) = \delta(n)$ to the system. This output sequence usually is denoted as $h(n)$.That is, if $x(n) = \delta(n)$, then $y(n) = h(n)$. Since any input sequence can be expressed as a weighted sum of unit impulses, knowing the unit impulse response enables us to determine the zero-state output for an arbitrary input.

3.2.1 Unit Impulse Response for Nonrecursive Systems

For a nonrecursive system described by the difference equation

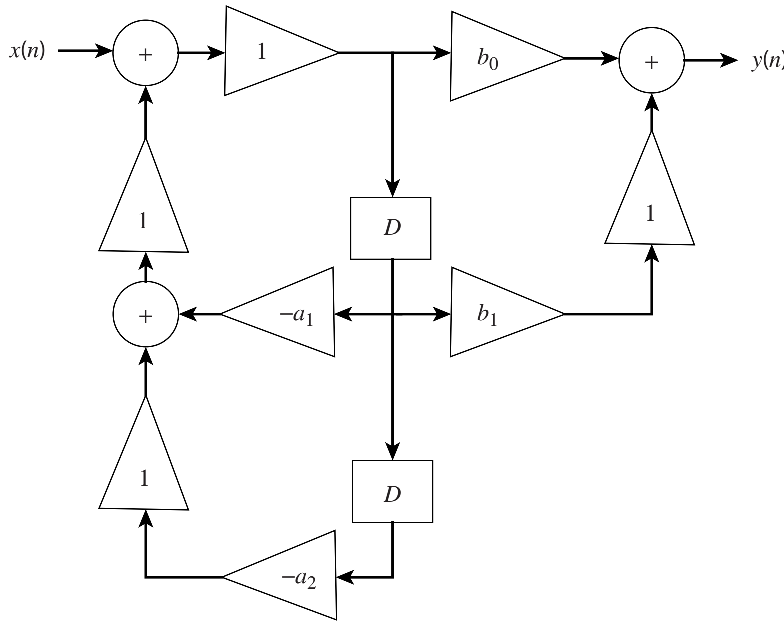
Nonrecursive system

$$y(n) = \sum_{k=0}^{L} b_k x(n-k)$$

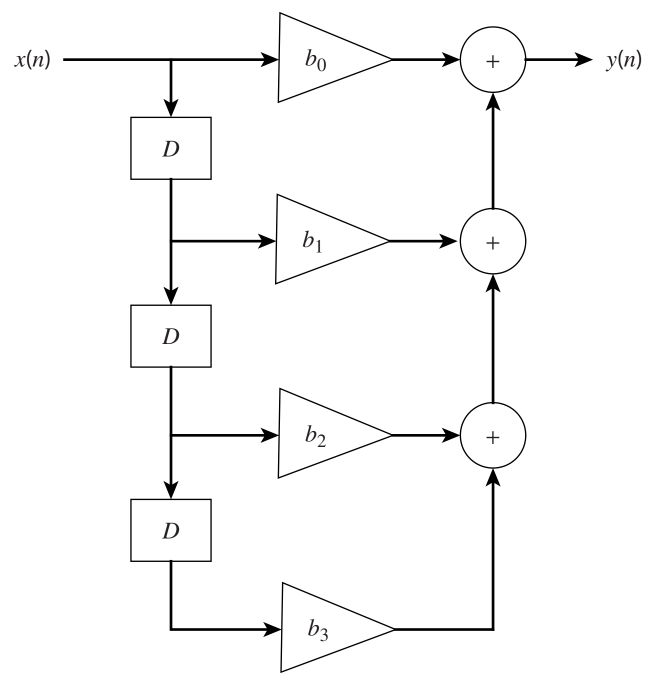
subjected to the unit impulse sequence of $x(n) = \delta(n)$, we see that the values of the output sequence are

$$\begin{aligned} y(n) &= 0, n < 0 \\ y(0) &= h(0) = b_0 \\ y(1) &= h(1) = b_1 \\ y(2) &= h(2) = b_2 \\ &\vdots \\ y(L) &= h(L) = b_L \\ y(L+1), y(L+2), \ldots &= 0. \end{aligned}$$

[1] Notice that simply by dividing all coefficients on both sides of the second form by a_0, we obtain an equation of the first form in which the coefficient of $y(n)$ is 1.



(a) Recursive system



(b) Nonrecursive system

Figure 3-2: Recursive and nonrecursive system diagrams

We notice that the response is zero for all $n < 0$ and $n > L$ and that the values of the unit impulse response $h(n)$ are the same as the values of the coefficients of the system's difference equation, namely,

$$y(n) = h(n) = \begin{cases} b_n, & 0 \le n \le L \\ 0, & n < 0 \text{ and } n > L. \end{cases}$$

This unit impulse response has a finite number of nonzero sample values, and thus, the system is defined as a *finite impulse response (FIR) system*. The general expression for the unit impulse response $h(n)$ for a FIR system is

> $h(n)$, *FIR system*
>
> $$h(n) = \sum_{k=0}^{L} b_k \delta(n-k) = \sum_{k=0}^{L} h(k)\delta(n-k).$$

3.2.2 Unit Impulse Response for Recursive Systems

To find the zero-state response to a unit impulse input for the causal N^{th}-order recursive system

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k)$$

we take the z-transform, setting all initial conditions to zero and yielding

$$\left[\sum_{k=0}^{N} a_k z^{-k}\right] Y(z) = \left[\sum_{k=0}^{L} b_k z^{-k}\right] X(z).$$

If $x(n) = \delta(n)$, $X(z) = 1$ and

$$Y(z) = \frac{\sum_{k=0}^{L} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

To obtain positive powers of z, we multiply by z^N/z^N

$$Y(z) = \frac{\sum_{k=0}^{L} b_k z^{N-k}}{\sum_{k=0}^{N} a_k z^{N-k}} = \frac{b_0 z^N + b_1 z^{N-1} + \ldots + b_L z^{N-L}}{a_0 z^N + a_1 z^{N-1} + \ldots + a_N}.$$

The relative sizes of N and L will be determined by the system design. There are three possibilities: $N < L$, $N = L$, and $N > L$, where each needs to be considered separately. So, quite arbitrarily, let's begin with $N = L$.To prepare for PFE, we divide $Y(z)$ by z. That results in

$$\begin{aligned}\frac{Y(z)}{z} &= \frac{b_0 z^N + b_1 z^{N-1} + \ldots + b_N}{z\left[a_0 z^N + a_1 z^{N-1} + \ldots + a_N\right]} \\ &= \frac{(1/a_0)\left[b_0 z^N + b_1 z^{N-1} + \ldots + b_N\right]}{z(z-d_1)(z-d_2)\ldots(z-d_N)}.\end{aligned}$$

Assuming distinct roots (poles) $d_1, d_2, \ldots, d_N$, we have the PFE

$$\frac{Y(z)}{z} = \frac{C_0}{z} + \frac{C_1}{z-d_1} + \frac{C_2}{z-d_2} + \ldots + \frac{C_N}{z-d_N}$$

where

$$\begin{aligned}C_0 &= \left.\frac{\cancel{z}(1/a_0)[b_0 z^N + b_1 z^{N-1} + \ldots + b_N]}{\cancel{z}(z-d_1)\ldots(z-d_k)\ldots(z-d_N)}\right|_{z=0} \\ &= \frac{b_N/a_0}{(-d_1)(-d_2)\ldots(-d_N)}.\end{aligned}$$

Notice that the factors z in both the numerator and the denominator cancel before substituting $z = 0$. The remaining PFE constants are found by the expression

$$\begin{aligned}C_k &= \left.\frac{\cancel{(z-d_k)}(1/a_0)\left[b_0 z^N + b_1 z^{N-1} + \ldots + b_N\right]}{(z-d_1)\ldots\cancel{(z-d_k)}\ldots(z-d_N)}\right|_{z=d_k} \\ &= \left.\frac{(1/a_0)\left[b_0 z^N + b_1 z^{N-1} + \ldots + b_N\right]}{(z-d_1)\ldots(z-d_{k-1})(z-d_{k+1})\ldots(z-d_N)}\right|_{z=d_k} \\ &= \frac{(1/a_0)\left[b_0 (d_k)^N + b_1 (d_k)^{N-1} + \ldots + b_N\right]}{(d_k-d_1)\ldots(d_k-d_{k-1})(d_k-d_{k+1})\ldots(d_k-d_N)},\end{aligned}$$
$$k = 1, 2, \ldots, N.$$

Where again we canceled the factors $(z - d_k)$ in both the numerator and the denominator before substituting $z = d_k$. Now we have

$$Y(z) = C_0 + \sum_{k=1}^{N} \frac{C_k z}{z - d_k} \quad \text{for} \quad N = L$$

and consequently, the unit impulse response is

$$y(n) = h(n) = C_0 \delta(n) + \sum_{k=1}^{N} C_k (d_k)^n u(n).$$

Thus, we define this result as an *infinite impulse response (IIR) system*, because the output will never reach zero—even an infinite number of samples after the input unit impulse sequence $\delta(n)$ becomes zero. That is, the terms $C_k(d_k)^n$, $k = 1, \ldots, N$ asymptotically approach but do not reach zero, even as $n \to \infty$.

Example 3-1: Unit Impulse Response of an IIR Filter with *N = L*

Given the first-order filter that has one input delay and one output delay $y(n) + a_1 y(n-1) = b_0 x(n) + b_1 x(n-1)$ with ICs $= 0$. Determine the unit impulse response $h(n)$.

Solution: Taking the z-transform of this DE yields

$$\left[1 + a_1 z^{-1}\right] Y(z) = \left[b_0 + b_1 z^{-1}\right] X(z).$$

For the unit impulse input $x(n) = \delta(n)$, the z-transform of the input is $X(z) = 1$, and the transform of the output $Y(z)$ is

$$Y(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} = \frac{b_0 z + b_1}{z + a_1}.$$

Expanding $Y(z)/z$ gives

$$\frac{Y(z)}{z} = \frac{b_0 z + b_1}{z(z + a_1)} = \frac{C_0}{z} + \frac{C_1}{z + a_1}$$

where $C_0 = b_1/a_1$ and $C_1 = (-b_0 a_1 + b_1)/(-a_1)$.

Returning to $Y(z)$ and taking the inverse transform produces the results

$$Y(z) = C_0 + \frac{C_1 z}{z + a_1}$$

and

$$y(n) = h(n) = C_0 \delta(n) + C_1(-a_1)^n u(n)$$

where C_0 and C_1 are as found previously. ■

Now let's consider the situation for $N > L$. From the earlier discussion,

$$Y(z) = \frac{b_0 z^N + b_1 z^{N-1} + \ldots + b_L z^{N-L}}{a_0 z^N + a_1 z^{N-1} + \ldots + a_N}$$

and to prepare for the PFE of $Y(z)/z$, we have

$$\begin{aligned}\frac{Y(z)}{z} &= \frac{b_0 z^{N-1} + b_1 z^{N-2} + \ldots + b_L z^{N-L-1}}{a_0 z^N + a_1 z^{N-1} + \ldots + a_N} \\ &= \frac{(1/a_0)\left[b_0 z^{N-1} + b_1 z^{N-2} + \ldots + b_L z^{N-L-1}\right]}{(z - d_1)(z - d_2)\ldots(z - d_N)}.\end{aligned}$$

Assuming distinct roots $d_1, d_2, \ldots, d_N$, the PFE of $Y(z)/z$ becomes

$$\frac{Y(z)}{z} = \frac{C_1}{z - d_1} + \frac{C_2}{z - d_2} + \ldots + \frac{C_N}{z - d_N}$$

where

$$C_k = \left.\frac{\cancel{(z - d_k)}(1/a_0)\left[b_0 z^{N-1} + b_1 z^{N-2} + \ldots + b_L z^{N-1-1}\right]}{(z - d_1)\ldots\cancel{(z - d_k)}\ldots(z - d_N)}\right|_{z = d_k}.$$

Consequently, after all the C_k for $k = 1, 2, \ldots, N$ are computed in the usual way, we have

$$Y(z) = \frac{C_1 z}{z - d_1} + \frac{C_2 z}{z - d_2} + \ldots + \frac{C_N z}{z - d_N}$$

and the unit impulse response sequence is given by

$$y(n) = h(n) = \sum_{k=1}^{N} C_k (d_k)^n u(n).$$

Example 3-2: Unit Impulse Response of an IIR Filter with *N > L*

Given the second-order IIR filter that has one delay of the input

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1)$$

with ICs = 0. Determine the unit impulse response $h(n)$.

Solution: The transformed DE with zero initial conditions is seen to be

$$\left[1 + a_1 z^{-1} + a_2 z^{-2}\right] Y(z) = \left[b_0 + b_1 z^{-1}\right] X(z)$$

and with $X(z) = 1$, the solution for the transformed output is

$$Y(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z}{z^2 + a_1 z + a_2}.$$

Assuming distinct denominator roots, the PFE of $Y(z)/z$ becomes

$$\begin{aligned}\frac{Y(z)}{z} &= \frac{b_0 z + b_1}{z^2 + a_1 z + a_2} = \frac{b_0 z + b_1}{(z - d_1)(z - d_2)} \\ &= \frac{C_1}{z - d_1} + \frac{C_2}{z - d_2}, d_1 \neq d_2\end{aligned}$$

where in terms of the literal difference-equation coefficients, the PFE coefficients are found to be

$$C_1 = (b_0 d_1 + b_1)/(d_1 - d_2), \quad C_2 = (b_0 d_2 + b_1)/(d_2 - d_1).$$

Thus

$$Y(z) = \frac{C_1 z}{z - d_1} + \frac{C_2 z}{z - d_2}$$

and

$$y(n) = h(n) = [C_1(d_1)^n + C_2(d_2)^n]u(n).$$ ■

We next consider an example of the situation where $N < L$.

Example 3-3: Unit Impulse Response for an IIR Filter with *N* < *L*

Find the unit impulse response $h(n)$ for the first-order system

$$y(n) - y(n-1) = x(n) + x(n-2)$$

Solution: Taking the z-transform with $X(z) = 1$ and zero initial conditions we have

$$\left[1 - z^{-1}\right] Y(z) = \left[1 + z^{-2}\right] \cdot 1$$

and

$$Y(z) = \frac{1 + z^{-2}}{1 - z^{-1}} = \frac{z^2 + 1}{z^2 - z}.$$

Then forming $Y(z)/z$ and expanding the result in partial fractions yields

$$\frac{Y(z)}{z} = \frac{(z^2+1)}{z^2(z-1)} = \frac{C}{z-1} + \frac{A_1}{z^2} + \frac{A_0}{z}$$

where

$$C = \left.\frac{z^2+1}{z^2}\right|_{z=1} = 2$$

$$A_1 = \left.\frac{z^2+1}{z-1}\right|_{z=0} = -1.$$

To find A_0, we first multiply $Y(z)/z$ by z as described in Section 2.6.5. **Inverse Transforms by Partial Fraction Expansion using Multiple Poles**. The result after substituting the values found for C and A_1 is

$$\frac{zY(z)}{z} = \frac{z(z^2+1)}{z^2(z-1)} = \frac{z \cdot 2}{z-1} + \frac{z \cdot (-1)}{z^2} + \frac{\cancel{z} \cdot A_0}{\cancel{z}}$$
$$= \frac{z^2+1}{z(z-1)} = \frac{2z}{z-1} + \frac{-1}{z} + A_0.$$

We then subtract from both sides the term $[-1/z]$ with the result

$$\frac{z^2+1}{z(z-1)} - \left[\frac{-1}{z}\right] = \frac{2z}{z-1} + \frac{-1}{z} + A_0 - \left[\frac{-1}{z}\right]$$
$$\frac{\cancel{z}(z+1)}{\cancel{z}(z-1)} = \frac{2z}{z-1} + A_0.$$

A_0 then can be evaluated in the usual way as

$$A_0 = \left.\frac{-z+1}{z-1}\right|_{z=0} = -1.$$

Now we have $Y(z) = [2z/(z-1) - z^{-1} - 1]$ yielding the unit impulse response $y(n) = 2u(n) - \delta(n-1) - \delta(n)$.

An m-file that evaluates and plots $y(n)$ follows, and the resulting output is shown in Figure 3-3. In looking at the m-file, you will notice that the constituents of $y(n)$ are formed by defining values of a step sequence of amplitude 2 and two unit impulse sequences in the interval $-5 \le n \le 10$. This can be a tedious process, but fortunately, we can simplify it greatly by using additional m-functions that we write. A revised m-file for this plot `F3_3WS`, that uses the California functions `ustpseqr` and `iseqr` is found on the FPAL website.

m-file

```
%F3_3 Unit impulse response for a first-order system
%Figure 3_3
n=-5:1:10;
yn1=[0 0 0 0 0 2 2 2 2 2 2 2 2 2 2 2];
yn2=[0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0];
yn3=[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0];
yn=yn1-yn2-yn3;
stem(n,yn);
xlabel ('sample number, n');
ylabel ('y(n)');
title ('Fig.3_3 Unit impulse response for a
   first-order system');
grid
axis([-5,10,-1, 3]);
```

■

Comment:

We leave development of the general form of the unit impulse response for $N \le L$ to the reader (see Problem 3.27). The result for this problem is given here.

For an N^{th}-order system with L delays of the input ($N \le L$) and with distinct characteristic roots $d_1, d_2, \ldots, d_N$ the unit impulse response $h(n)$ is given by

$$\boxed{h(n) = \sum_{k=1}^{N} C_k (d_k)^n + \sum_{k=0}^{L-N} A_k \delta(n-k).}$$

We observe that if $N > L$ the upper limit of the second summation is negative and these terms are absent. In this case, the form simplifies to our earlier result. In summary then, the equation given here is valid for the general case, regardless of the relative sizes of N and L.

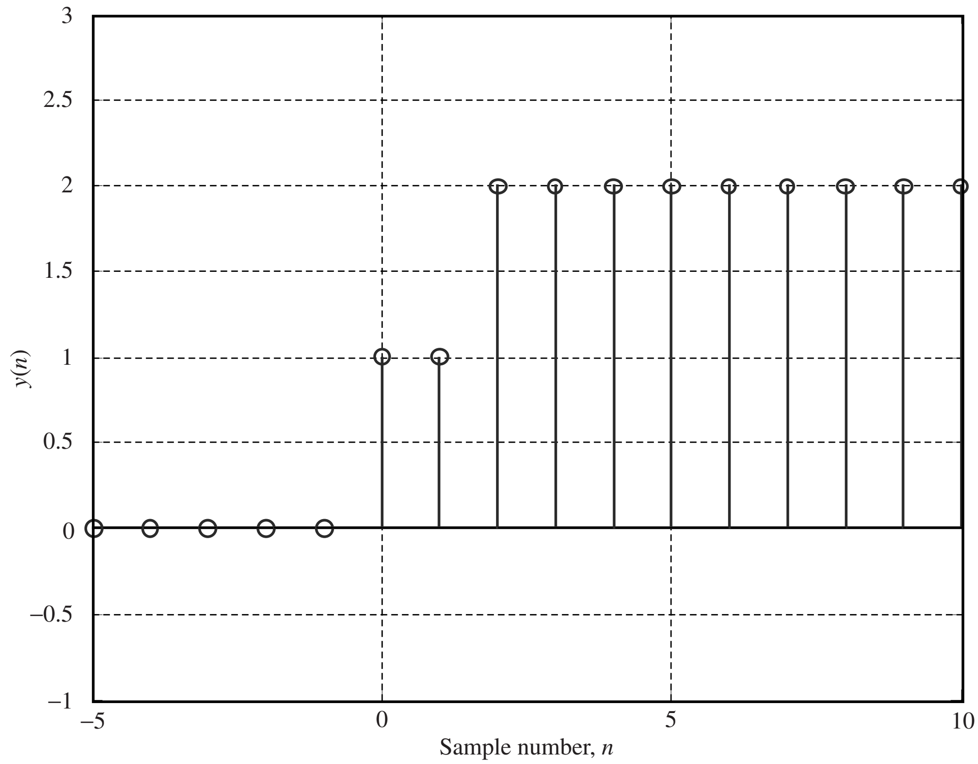


Figure 3-3: Unit impulse response for a first-order system

3.3 Transfer Functions

The transfer function of a linear, time-invariant, discrete-time system is defined in terms of the output and input sequences as

Definition

$$\text{Transfer function} = \frac{z\text{-transform of the zero-state output sequence } y_{zs}(n)}{z\text{-transform of the input sequence } x(n)}$$

where the initial conditions are zero. In terms of symbols,

Symbols

$$H(z) = \frac{Y(z)}{X(z)}.$$

Alternatively, starting with the system's difference equation,

Difference equation (DE)

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k)$$

and using the shifting property and linearity with zero initial conditions gives the algebraic z-domain equation

$$a_0 Y(z) + a_1 z^{-1} Y(z) + \cdots + a_N z^{-N} Y(z) =$$
$$b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_L z^{-L} X(z)$$

which can be written in terms of two summations as

Transformed DE

$$Y(z)\left[\sum_{k=0}^{N} a_k z^{-k}\right] = X(z)\left[\sum_{k=0}^{L} b_k z^{-k}\right].$$

Solving for $H(z) = Y(z)/X(z)$, we find the transfer function to be

Rational function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{L} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

which is the ratio of two polynomials—a rational function. We notice that the coefficients of the input terms in the difference equation are the coefficients of the numerator polynomial of the transfer function and that the coefficients of the output terms in the difference equation are the coefficients of the denominator polynomial. Recall that a delay of Q samples in a difference equation corresponds to multiplication by z^{-Q} in the transformed difference equation.

Example 3-4: Transfer Function from a Difference Equation

Given the difference equation for a causal system

$$\begin{aligned} &y(n) - 4y(n-1) + 6y(n-2) - 4y(n-3) \\ &\quad = x(n) + x(n-5) \end{aligned}$$

substitute the appropriate values of the system's difference-equation coefficients in the general formula for the transfer function to determine the transfer function $H(z)$ for this system.

Solution: We see from the DE that $a_0 = 1$, $a_1 = -4$, $a_2 = 6$, $a_3 = -4$, $b_0 = 1$, and $b_5 = 1$. Thus, the rational function $H(z)$ is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{L} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{1 + z^{-5}}{1 - 4z^{-1} + 6z^{-2} - 4z^{-3}}.$$

■

An Important Connection:

If a unit impulse sequence $\delta(n)$ is the input to an LTI system, the transform of the input is $X(z) = 1$ and $Y(z) = H(z) \cdot X(z) = H(z)$. Taking the inverse z-transform gives $y(n) = h(n)$. Thus, the transfer function of an LTI system can be found by taking the z-transform of the system's unit impulse response $h(n)$ as $h(n) \Leftrightarrow H(z)$, so

$$H(z) = \mathbb{Z}[h(n)].$$

Conversely, the unit impulse response $h(n)$ can be found by evaluating the inverse z-transform of the transfer function $H(z)$, so that

$$h(n) = \mathbb{Z}^{-1}[H(z)].$$

Example 3-5: Transfer Function from a Unit Impulse Response

For the system whose unit impulse response is

$$h(n) = [(0.5)^n + (-0.5)^n]u(n)$$

find the transfer function model $H(z)$.

Solution: Using linearity and transform pair 2 from Table 2.1 for an exponential sequence gives

$$\begin{aligned} H(z) &= \mathbb{Z}\left[(0.5)^n u(n) + (-0.5)^n u(n)\right] \\ &= \frac{z}{z-0.5} + \frac{z}{z+0.5} = \frac{2z^2}{z^2 - 0.25} \end{aligned}$$

where the region of convergence for each component of $H(z)$ is $|z| > 0.5$, so the region of convergence for $H(z)$ is also $0.5 < |z|$. ■

3.3.1 Poles and Zeros

A transfer function can be factored into first-order terms as follows:

$$\begin{aligned} H(z) &= \frac{b_0(1 - n_1 z^{-1})(1 - n_2 z^{-1}) \cdots (1 - n_L z^{-1})}{a_0(1 - d_1 z^{-1})(1 - d_2 z^{-1}) \cdots (1 - d_N z^{-1})} \\ &= \frac{b_0 z^{N-L}(z - n_1)(z - n_2) \cdots (z - n_L)}{a_0(z - d_1)(z - d_2) \cdots (z - d_N)}. \end{aligned}$$

As noted in Chapter 2, the values of z that make $H(z)$ go to zero and infinity are called the *system zeros* and *system poles*, respectively. There are zeros at $z = n_1, n_2, \ldots, n_L$ and poles at $z = d_1, d_2, \ldots, d_N$. The term z^{N-L} governs the transfer function characteristics at $z = 0$. For $N - L > 0$, there are $N - L$ zeros at $z = 0$; for $N - L < 0$, there are $L - N$ poles at $z = 0$; for $N = L$, the factor z^{N-L} is absent.

3.3.2 Region of Convergence

The region of convergence must be specified for any transform, and the transfer function is not excepted from this rule. A transfer function is the z-transform of the unit impulse response, so the observations made in Section 2.5.2(e) apply also to $H(z)$.

3.3.3 Linear Convolution

In Chapter 2, "Linear Discrete-Time Systems and z-Transforms," the zero-state response (ICs equal to zero) of an LTI system was obtained from the convolution sum

$$\begin{aligned} y(n) &= h(n) * x(n) = x(n) * h(n) \\ &= \sum_{m=-\infty}^{\infty} h(m)x(n-m) = \sum_{m=-\infty}^{\infty} x(m)h(n-m) \end{aligned}$$

where $h(n)$ is the unit impulse response of the system and $x(n)$ represents the system input sequence. We assume that the system is causal, so $h(n) = 0$ and the input is zero for $n < 0$, a causal sequence. With these assumptions, the two versions of the convolution sum become

$$y(n) = \sum_{m=0}^{n} h(m)x(n-m) = \sum_{m=0}^{n} h(n-m)x(m).$$

Earlier in this chapter, the system transfer function was defined as $H(z) = Y(z)/X(z)$. If the z-transform $X(z)$ of the input $x(n)$ is known, the z-transform $Y(z)$ of the output sequence $y(n)$ is

$$Y(z) = H(z) \cdot X(z) = X(z) \cdot H(z)$$

and the output $y(n)$ can be found by the inverse z-transform.

Thus, we have the convolution pair or property that allows the evaluation of a convolution in the time domain either directly or indirectly by the means of z-transforms:

$$\begin{aligned} y(n) &= \mathbb{Z}^{-1}[Y(z)] \\ &= \mathbb{Z}^{-1}[H(z) \cdot X(z)] = \mathbb{Z}^{-1}[X(z) \cdot H(z)] \\ &= h(n) * x(n) = x(n) * h(n). \end{aligned}$$

Linear convolution

$$\begin{aligned} y(n) &= \sum_{m=-\infty}^{\infty} h(m)x(n-m) \\ &= \sum_{m=-\infty}^{\infty} h(n-m)x(m) \Leftrightarrow H(z) \cdot X(z) \end{aligned}$$

where the region of convergence for $H(z) \cdot X(z)$ will be determined as the intersection of the ROCs of $H(z)$ and $X(z)$.

Example 3-6: Two Methods of Evaluating Linear Convolution

A lowpass digital filter with the unit impulse response

$$h(n) = [(0.9)^n - (0.8)^n]u(n)$$

is subjected to a unit step input, that is, $x(n) = u(n)$.

(a) Use z-transforms to find the filter's output $y(n) = h(n) * x(n)$.

(b) Verify your answer to part a by using the "time-domain" convolution sum.

Solution:

(a) We use the convolution pair or property $Y(z) = H(z) \cdot X(z)$. From Table 2.2,

$$\begin{aligned} H(z) &= \frac{z}{z-0.9} - \frac{z}{z-0.8} \\ &= \frac{0.1z}{(z-0.9)(z-0.8)}, \quad |z| > 0.9 \end{aligned}$$

and

$$X(z) = \frac{z}{z-1}, \quad |z| > 1.$$

Thus, the output transform is

$$Y(z) = \frac{0.1z}{(z-0.9)(z-0.8)} \cdot \frac{z}{z-1}$$

that has the partial fraction expansion

$$Y(z) = \frac{5z}{z-1} + \frac{4z}{z-0.8} - \frac{9z}{z-0.9}, \quad |z| > 1$$

and table lookup gives the output sequence

$$y(n) = [5 + 4(0.8)^n - 9(0.9)^n]u(n).$$

(b) Using the convolution sum and the closed form expression for the finite geometric series (see Appendix A) we have

$$\begin{aligned} y(n) &= \sum_{0}^{n}(0.9)^m - \sum_{0}^{n}(0.8)^m \\ &= \frac{1-(0.9)^{n+1}}{1-0.9} - \frac{1-(0.8)^{n+1}}{1-0.8} \\ &= \frac{1}{0.1} - \frac{0.9(0.9)^n}{0.1} - \frac{1}{0.2} + \frac{0.8(0.8)^n}{0.2} \\ &= 5 - 9(0.9)^n + 4(0.8)^n \end{aligned}$$

for $n \geq 0$.

Notice that although we were able to easily evaluate the convolution summation here and obtain an analytical result, this frequently is difficult to do. What made it possible in this case is that the input sequence $x(n)$ was either 1 or 0 for all values of n. Generally, we will find it easier to find a system output using z-transforms or a numerical procedure. ■

Comment:

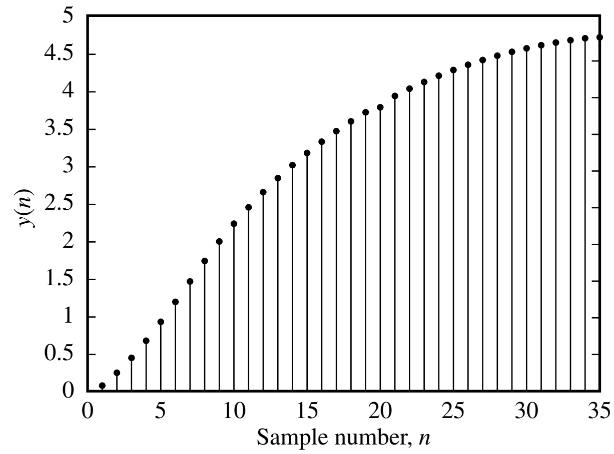
There are several m-functions that can be used to obtain a graphical solution to this problem. Two of them are `filter` and `dstep`. Recall that the transfer function is

$$H(z) = \frac{0.1z}{z^2 - 1.7z + 0.72}.$$

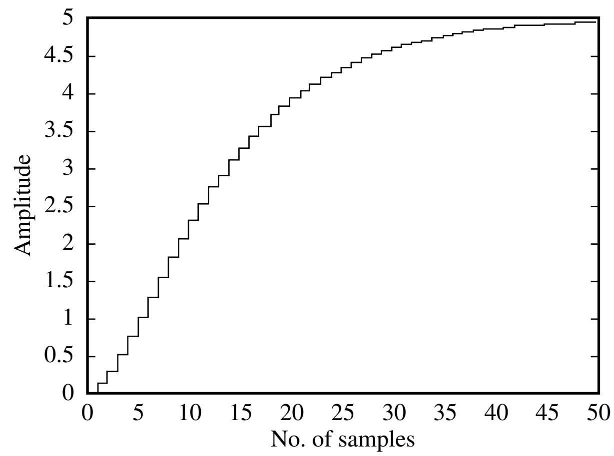
The m-file using `filter` is next, and the plot is shown in Figure 3-4a. The m-file for Figure 3-4b uses the transfer-function input to the function `dstep` and the option that plots automatically (if available in mathscript).

```
                    m-file
%F3_4 System output using filter and dstep
%F3_4a Unit step response from filter
n=0:1:35; % evaluate for 36 samples
b=[0, .1,0]; % coefficients of input terms, the x(n)'s
a=[1,-1.7,.72]; % coefficients of output terms, the y(n)'s
x=[1*ones(size(n))]; % constant input of 1
y=filter(b,a,x); % call filter
%...plotting statements

pause
%F3_4b Unit step response from dstep
num= [0, . 1, 0]; % numerator coefficients
den=[1,-1.7,0.72]; % denominator coefficients
dstep(num,den); % call dstep
```



(a) From filter



(b) From dstep

Figure 3-4: Unit step response

3.3.4 Sinusoidal Steady-State Response and Frequency Response

A very important and practical situation occurs when a stable causal system described by the transfer function $H(z) = Y(z)/X(z)$ is subjected to the sinusoidal input sequence $x(n) = A\cos(n\theta)u(n)$ where A is the amplitude of the sinusoid and θ is the digital frequency. Let us show that the system's steady-state output response $y_{ss}(n) = \lim_{n\to\infty} y(n)$ is given by the expression

$$y_{ss}(n) = A|H(e^{j\theta})|\cos[n\theta + \angle H(e^{j\theta})]$$

where

$$H(e^{j\theta}) = H(z)|_{z=e^{j\theta}} = |H(e^{j\theta})|e^{j\angle H(e^{j\theta})}.$$

From Table 2.1, the z-transform of the output caused by the input $x(n) = A\cos(n\theta)\,u(n)$ is

$$Y(z) = H(z)\cdot X(z) = H(z)\cdot\left\{\frac{Az(z-\cos\theta)}{(z-e^{j\theta})(z-e^{-j\theta})}\right\},$$

and for a recursive system

$$Y(z) = \frac{b_0 z^{N-L}(z-n_1)\ldots(z-n_L)}{a_0(z-d_1)\ldots(z-d_N)}\left[\frac{Az(z-\cos\theta)}{(z-e^{j\theta})(z-e^{-j\theta})}\right].$$

The first move is to obtain the partial fraction expansion for $Y(z)$.

Assuming nonrepeated (distinct) poles for $H(z)$ and expanding in partial fractions produces

PFE

$$Y(z) = \sum_{k=0}^{L-N} A_k z^{-k} + \frac{C_1 z}{z-d_1} + \frac{C_2 z}{z-d_2} + \cdots + \frac{C_N z}{z-d_N} + \frac{C_{\theta 1} z}{z-e^{j\theta}} + \frac{C_{\theta 2} z}{z-e^{-j\theta}}$$

where $C_{\theta 1}$ and $C_{\theta 2}$ are the PF constants for the input terms. Taking the inverse transform yields the solution for $n \geq 0$:

$$y(n) = \sum_{k=0}^{L-N} A_k \delta(n-k) + \Big[C_1(d_1)^n + C_2(d_2)^n + \cdots + C_N(d_N)^n + C_{\theta 1} e^{jn\theta} + C_{\theta 2}\, e^{-jn\theta} \Big] u(n).$$

Since the system is *stable*, all terms except $C_{\theta 1} e^{jn\theta}$ and $C_{\theta 2} e^{-jn\theta}$ will approach zero as $n \to \infty$ (because for a stable, causal system, $|d_k| < 1, k = 0, 1, \ldots, N$), leaving only the steady-state solution

$$y_{ss}(n) = C_{\theta 1} e^{jn\theta} + C_{\theta 2} e^{-jn\theta}$$

that in the z-transform domain can be written as

$$Y_{ss}(z) = \frac{C_{\theta 1} z}{z - e^{j\theta}} + \frac{C_{\theta 2} z}{z - e^{-j\theta}}.$$

Recalling that the z-transform of the output sequence is

$$Y(z) = H(z) \cdot \left\{ \frac{Az(z - \cos\theta)}{(z - e^{j\theta})(z - e^{-j\theta})} \right\}$$

we then divide both sides of the equation by z in order to find the partial fraction constants, giving

$$\frac{Y(z)}{z} = \frac{1}{\cancel{z}} H(z) \cdot \left\{ \frac{A\cancel{z}(z - \cos\theta)}{(z - e^{j\theta})(z - e^{-j\theta})} \right\}$$

where the z-factors in the numerator and denominator of the right side of the equation are cancelled.

As already demonstrated, only the PFE coefficients $C_{\theta 1}$ and $C_{\theta 2}$ are of interest for the steady-state solution. The partial fraction constant $C_{\theta 1}$ (usually a complex number) is found from

PF constant

$$C_{\theta 1} = \left[H(z) \cdot \left\{ \frac{A(z - \cos\theta)}{\cancel{(z - e^{j\theta})}(z - e^{-j\theta})} \right\} \cdot \cancel{(z - e^{j\theta})} \right] \Bigg|_{z = e^{j\theta}}$$

$$= H(e^{j\theta}) \cdot \left\{ \frac{A\left(e^{j\theta} - \dfrac{e^{j\theta} + e^{-j\theta}}{2} \right)}{\left(e^{j\theta} - e^{-j\theta} \right)} \right\}$$

$$= \frac{A}{2} H(e^{j\theta}) = \frac{A}{2} \left| H(e^{j\theta}) \right| e^{j\angle H(e^{j\theta})}$$

where $|H(e^{j\theta})|$ is the magnitude of the complex number $H(e^{j\theta})$ and $\angle H(e^{j\theta})$ is its phase in radians. In a similar manner, the other PFE constant is found from

$$C_{\theta 2} = \left[H(z) \cdot \left\{ \frac{A(z - \cos\theta)}{(z - e^{j\theta})\cancel{(z - e^{-j\theta})}} \right\} \cdot \cancel{(z - e^{j\theta})} \right] \Bigg|_{z = e^{-j\theta}}$$

$$= H(e^{-j\theta}) \cdot \left\{ \frac{A\left(e^{-j\theta} - \dfrac{e^{j\theta} + e^{-j\theta}}{2} \right)}{\left(e^{-j\theta} - e^{j\theta} \right)} \right\}$$

$$= \frac{A}{2} H(e^{-j\theta}) = \frac{A}{2} \left| H(e^{-j\theta}) \right| e^{j\angle H(e^{-j\theta})}.$$

We notice that $C_{\theta 1}$ and $C_{\theta 2}$ are complex conjugates of one another because $H(e^{j\theta})$ and $H(e^{-j\theta})$ are complex conjugates; $H(e^{-j\theta})$ is found from $H(e^{j\theta})$ by replacing j everywhere it appears with $-j$. Therefore,

$$|H(e^{j\theta})| = |H(e^{-j\theta})| \quad \text{and} \quad \angle H(e^{j\theta}) = -\angle H(e^{-j\theta})$$

or collectively,

$$H(e^{j\theta}) = |H(e^{j\theta})| e^{j\angle H(e^{j\theta})}$$

and

$$H(e^{-j\theta}) = |H(e^{j\theta})| e^{-j\angle H(e^{j\theta})}$$

which can be written as

$$H(e^{j\theta}) = H^*(e^{-j\theta})$$

where the superscript $*$ denotes a complex conjugate. Now we can write the steady-state response in the form

$$y_{ss}(n) = \left\{ \frac{A}{2} |H(e^{j\theta})| e^{j\angle H(e^{j\theta})} \right\} \cdot e^{jn\theta} + \left\{ \frac{A}{2} |H(e^{j\theta})| e^{-j\angle H(e^{j\theta})} \right\} \cdot e^{-jn\theta}$$

$$= A|H(e^{j\theta})| \left\{ \frac{e^{j[n\theta + \angle H(e^{j\theta})]} + e^{-j[n\theta + \angle H(e^{j\theta})]}}{2} \right\}$$

$$= A|H(e^{j\theta})| \cdot \cos(n\theta + \angle H(e^{j\theta}))$$

which is the result we set out to derive. We observe that this is the same result derived by using superposition in Section 2.4.

This equation makes the *evaluation* of a discrete-time system's steady-state response to a sinusoidal input sequence straightforward, and we refer to the result as the sinusoidal steady-state formula. In the same way, it can be shown that if the input to a stable, causal LTI, discrete-time system is a sinusoid with a phase displacement of β rad, that is,

$$x(n) = A\cos(n\theta + \beta)\, u(n)$$

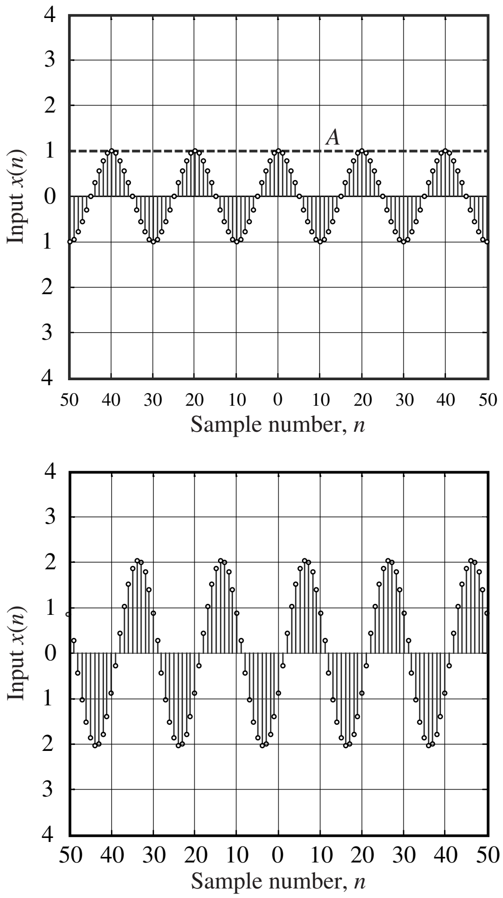


Figure 3-5: Input sequence (*left*); output sequence (*right*)

the steady-state output is

$$y_{ss}(n) = A|H(e^{j\theta})| \cdot \cos(n\theta + \beta + \angle H(e^{j\theta})).$$

Thus, when a sinusoidal input sequence $x(n) = A\cos(n\theta + \beta)u(n)$ is applied to a *stable* LTI system, then the steady-state output $y_{ss}(n)$ has the characteristics:

1. Its amplitude is the input amplitude A multiplied by the gain $|H(e^{j\theta})|$.
2. Its phase is the input phase β shifted by the angle $\angle H(e^{j\theta})$.

This is illustrated in Figure 3-5, where the input sequence and the steady-state output sequence of a stable LTI system subjected to a sinusoidal input are shown.

Important Comment:

The quantity

$$H(e^{j\theta}) = H(z)\Big|_{z=e^{j\theta}} = \Big|H(e^{j\theta})\Big|\, e^{j\angle H(e^{j\theta})}$$

which is known as the system's *frequency response*, plays an important role in the analysis and design of discrete-time systems. Indeed, many design problems are described in terms of the desirable characteristics of a system's frequency response. Once a system's frequency response is known, the effect of the system on *all* input frequencies can be determined. Chapter 4 contains a comprehensive treatment of the frequency response of discrete-time systems.

Example 3-7: Using the Sinusoidal Steady-State Formula

A digital notch filter designed to block the frequency $\pi/2$ can be modeled by $y(n) + 0.90y(n-2) = x(n) + x(n-2)$, where $y(n)$ represents the filter output and $x(n)$ is its input. The input sequence $x(n) = A + B\cos(n\pi/2)$ is applied to this filter. Find the equation for the steady-state output voltage $y_{ss}(n)$.

Solution: Following the procedure just outlined, we first find the transfer function

$$H(z) = \frac{1 + z^{-2}}{1 + 0.9z^{-2}} = \frac{z^2 + 1}{z^2 + 0.9}.$$

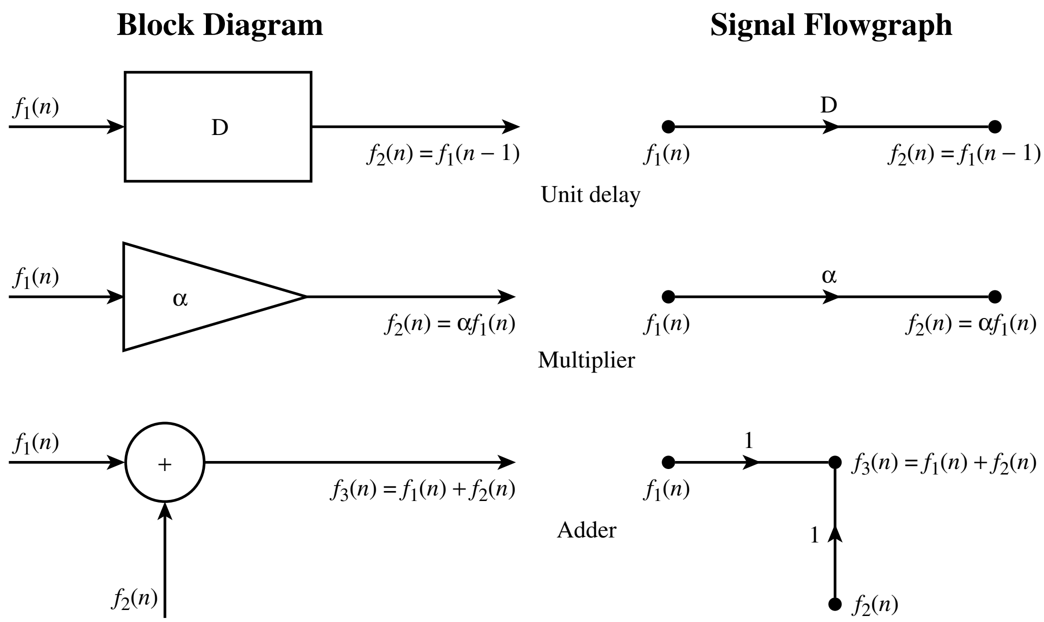
Next, $H(z)$ needs to be evaluated at the frequencies of the input sequence, which are $\theta = 0$ (a constant such as A can be considered a sinusoid of zero frequency) where $z = e^{j0} = 1$ and $\theta = \pi/2$ or $z = e^{j\pi/2} = j1$. Thus, we have

$$H\,(1) = \frac{1^2 + 1}{1^2 + 0.9} = 1.053 \text{ and } H(j1) = \frac{(j1)^2 + 1}{(j1)^2 + 0.9} = 0.$$

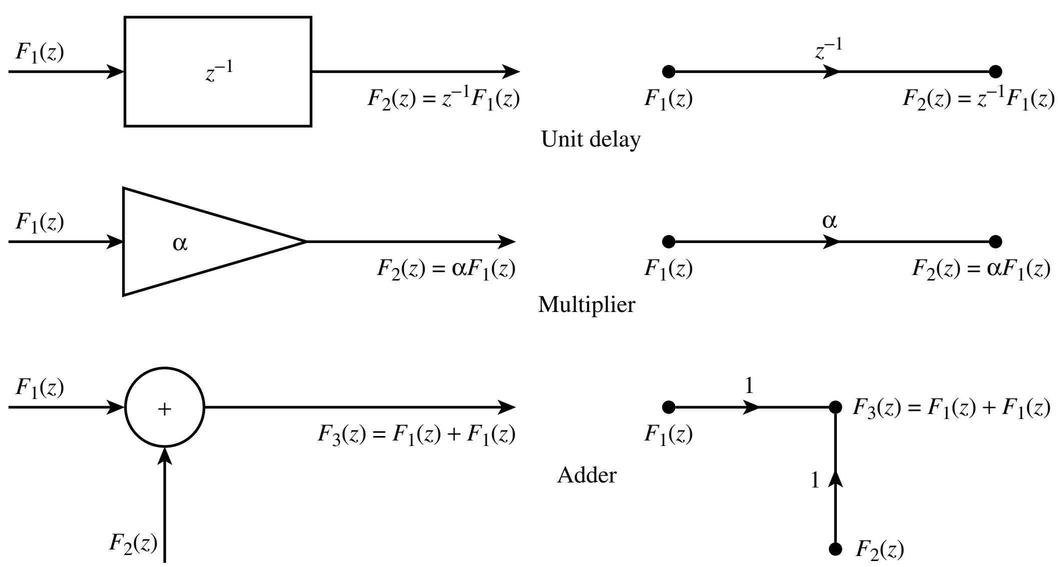
Finally, applying the sinusoidal steady-state formula gives $y_{ss}(n) = 1.053A$, and the filter does indeed notch out (or suppress) the frequency of $\theta = \pi/2$. ■

3.4 System Diagrams or Structures

We have modeled linear, time-invariant, discrete-time systems by constant-coefficient difference equations, by unit impulse responses, and by transfer functions. Although two simple structures of recursive and nonrecursive systems were depicted in Figure 3-2, in this section we discuss models in the form of system diagrams, a graphical way of representing the same information contained in the previous models. Such diagrams can provide useful visualizations of system structure and insight into system implementation using digital hardware or software algorithms. We also introduce the graphical algorithm of Mason's Gain Rule that enables us to find transfer functions from a system diagram.



(a) System diagram symbols in the time domain



(b) System diagram symbols in the z-domain

Figure 3-6: System diagram symbols

3.4.1 Symbols for System Diagrams

Recall that when the general form of a system's difference equation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k)$$

is solved for $y(n)$, we obtain

$$y(n) = \frac{1}{a_0}\left[-\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{L} b_k x(n-k)\right]$$

where we notice that delays, multipliers, and adders are required to implement this equation. Figure 3-6a and b shows two sets of commonly used symbols both in the time and the z domains

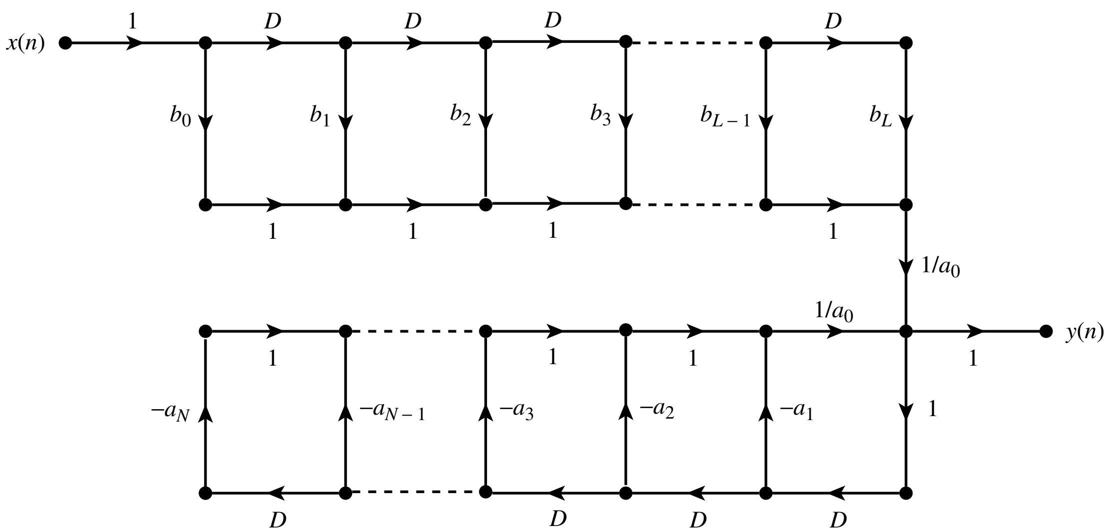


Figure 3-7: System diagram for a DE

as well as the describing relations. Normally, these sets of symbols are not mixed, and it is strictly a matter of preference as to the use of signal flowgraphs (SFGs) or block diagrams. We will use both.

The three building blocks in Figure 3-6a are readily implemented with digital hardware and software. For example, unit-delay elements can be implemented using shift registers. Multipliers and adders can be realized with off-the-shelf digital hardware. The pervasiveness of digital signal processing systems has led, however, to the development of special-purpose digital hardware architectures, software, and product development tools by companies such as Texas Instruments (see www.ti.com). Alternatively, a discrete-time system also can be implemented by a software algorithm executed on a general-purpose computing platform.

3.4.2 Construction of System Diagrams

In Figure 3-7, a diagram that represents the general difference equation is shown where we notice that the top half of the diagram represents (realizes) the input or nonrecursive terms of the difference equation, and the bottom half shows the output or recursive terms.

Example 3-8: Drawing a System Diagram

A bandpass digital filter is represented by the recursive difference equation

$$y(n) = -0.81y(n-2) + 0.105x(n) - 0.085x(n-2).$$

Draw a signal flowgraph in the manner of Figure 3-7 or a block diagram that represents this filter.

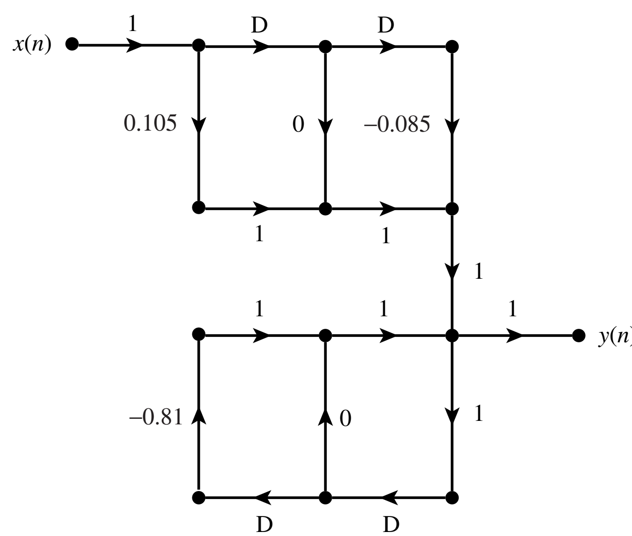


Figure 3-8: System diagram for Example 3-8

Solution: Comparing the filter's difference equation with that of the general form of the difference equation, we see that $a_0 = 1, a_1 = 0, a_2 = 0.81, b_0 = 0.105, b_1 = 0,$ and $b_2 = -0.085$. The system diagram marked with these gains is given in Figure 3-8. The multipliers showing zero gain could be eliminated, but this is simply a matter of choice. ■

The general system diagram from the z-transform point of view is shown in Figure 3-9. There we have replaced $x(n)$ with its transform $X(z)$, $y(n)$ with its transform $Y(z)$, the unit delays of D with the transform equivalent z^{-1}, and a delayed input $x(n-k)$ and output $y(n-k)$ with $z^{-k}X(z)$ and $z^{-k}Y(z)$,

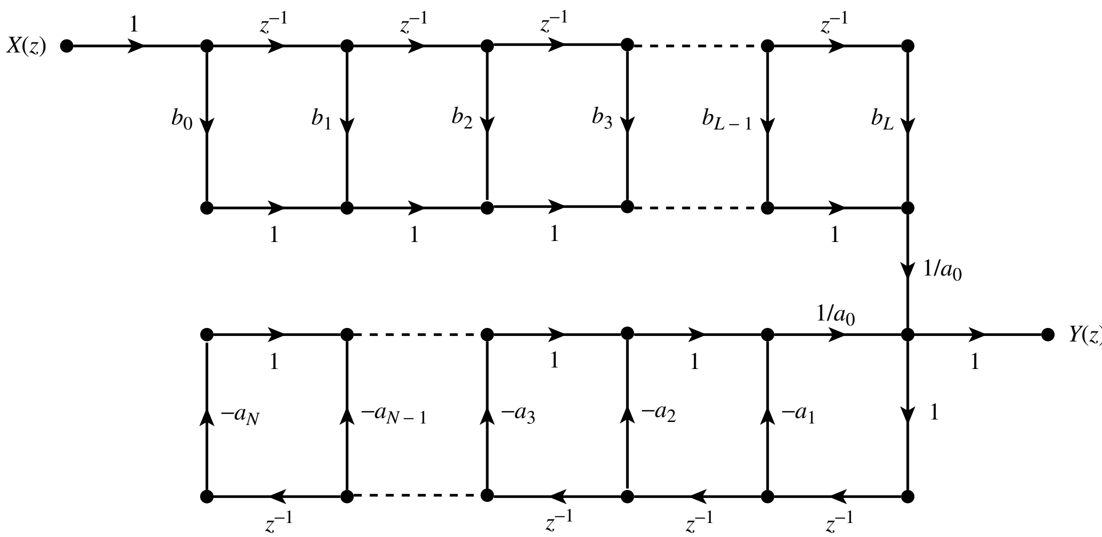


Figure 3-9: System diagram for a transformed DE

respectively. Consequently, the system's difference equation becomes the algebraic equation

$$Y(z) = \frac{1}{a_0}\left[-\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{L} b_k z^{-k} X(z)\right].$$

A system diagram also can be generated from the transfer function as the product of two transfer functions as follows:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{L} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \cdot \frac{\sum_{k=0}^{L} b_k z^{-k}}{1}$$

where $a_0 = 1$ is assumed. Next, we define a new variable $Q(z)$ such that

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{Q(z)}{X(z)} \cdot \frac{Y(z)}{Q(z)} \\ &= \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \cdot \frac{\sum_{k=0}^{L} b_k z^{-k}}{1} \\ &= H_1(z) \cdot H_2(z) \end{aligned}$$

where

$$H_1(z) = \frac{Q(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

and

$$H_2(z) = \frac{Y(z)}{Q(z)} = \frac{\sum_{k=0}^{L} b_k z^{-k}}{1}.$$

It is important to notice that $H_1(z)$ represents the recursive portion of the overall transfer function $H(z)$, while $H_2(z)$ represents the nonrecursive portion of $H(z)$. Alternatively, we can think of $H_1(z)$ as implementing the poles of $H(z)$ and $H_2(z)$ as implementing the zeros of $H(z)$. Notice also that we can write $H(z)$ as

$$H(z) = H_1(z)H_2(z) = H_2(z)H_1(z)$$

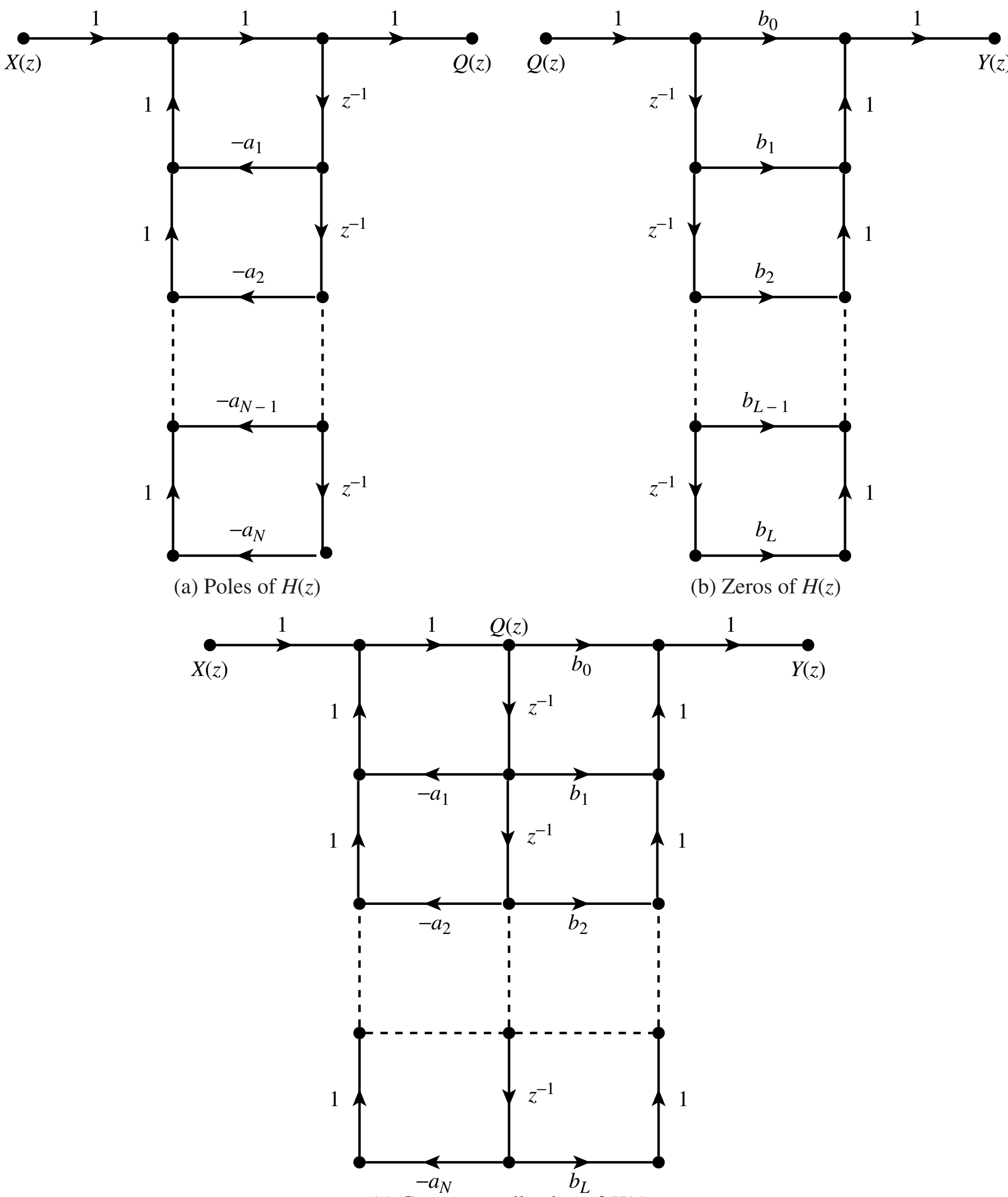
and that the stability of the overall system will be determined by $H_1(z)$, because it is the poles that control this important characteristic.

Rearranging the equation for $H_1(z)$ and taking the inverse transform gives

$$Q(z) = X(z) - \sum_{k=1}^{N} a_k z^{-k} Q(z)$$

and

$$q(n) = x(n) - \sum_{k=1}^{N} a_k q(n-k).$$



(c) Compact realization of $H(z)$

Figure 3-10: Realization of a transfer function

See Figure 3-10a for the implementation of the equation for $Q(z)$ that realizes the poles of the system. Finally, rearranging the equation for $H_2(z)$ and taking the inverse transform gives the output relations

$$Y(z) = \sum_{k=0}^{L} b_k z^{-k} Q(z) \quad \text{and} \quad y(n) = \sum_{k=0}^{L} b_k q(n-k).$$

In Figure 3-10b we have shown the implementation of this output equation that realizes the system zeros; notice that this diagram is the general form for a nonrecursive system. Figure 3-10c (which assumes $N = L$) portrays the combination of the output and the input sections that share the delay elements to give a general diagram for a recursive system.

Example 3-9: System Diagram from a Transfer Function

A digital filter is described by the fourth-order transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-2} + z^{-4}}{1 - z^{-1} - 0.31z^{-2} + 0.81z^{-3} - 0.405z^{-4}}.$$

Draw a transform domain SFG for the recursive (pole generating) transfer function

$$H_1(z) = \frac{Q(z)}{X(z)} = \frac{1}{1 - z^{-1} - 0.31z^{-2} + 0.81z^{-3} - 0.405z^{-4}}$$

and for the nonrecursive (zero generating) transfer function

$$H_2(z) = \frac{Y(z)}{Q(z)} = \frac{1 - 2z^{-2} + z^{-4}}{1}.$$

Connect the two together for the overall system SFG.

Solution: In Figure 3-11a, we have implemented the algebraic equation

$$Q(z) = X(z) + Q(z)\left[z^{-1} + 0.31z^{-2} - 0.81z^{-3} + 0.405z^{-4}\right]$$

and in Figure 3-11b we have

$$Y(z) = Q(z)\left[1 - 2z^{-2} + z^{-4}\right].$$

The composite diagram is shown in Figure 3-11c on the following page. ■

3.4.3 Mason's Gain Rule and Applications

In the early 1950s as part of his doctoral studies at the Massachusetts Institute of Technology, Professor Samuel J. Mason developed a rule (or algorithm) that was based on a graphical procedure for calculating transfer functions between input and output points of a signal flowgraph or an equivalent block diagram. The proof of this rule is based on Cramer's method for solving a set of simultaneous algebraic equations; consequently, we can apply the Mason Gain Rule (MGR) in the algebraic z-domain. We explain the rule by using the SFG of Figure 3-12a on page 21, which models a sampled-data feedback control system.

Let us consider the following general definitions and their applicability to this signal flowgraph.

Nodes. Points on a graph where the signals appear. All the heavy dots are nodes, with the most important of these being $X(z)$, $E(z)$, $\Theta(z)$, and $\Omega(z)$. At any node, the quantities associated with the incoming branches are summed, whereas the outgoing branches have no effect on the signal at the node; that is, we don't apply Kirchhoff's current law at the node.

Branch. A directed line segment, having an associated gain, that connects two nodes. An unmarked branch is assumed to have a gain of 1. Two easily identified branches are the delays with gains of z^{-1}.

Input node. Has no incoming branches. Obviously, $X(z)$ is the only input node.

Output node. Must have at least one incoming branch. All the rest of the nodes of the graph are output nodes. We use the node $\Theta(z) = Y(z)$ as the designated system output.

Path. A continuous sequence of branches, traversed in the indicated branch direction, along which no node is encountered more than once. From $X(z)$ to $Y(z) = \Theta(z)$, there are two paths (P_1 and P_2), as indicated in Figure 3-12b.

Loop. A continuous sequence of branches traversed in the indicated branch directions from one node around a closed path back to the same node along which no node is encountered more than once. This graph has four loops: α, β, γ, and δ, as indicated in Figure 3-12c.

Mason's Gain Rule (MGR) is

Mason's Gain Rule

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=1}^{R} P_k(z) \cdot \Delta_k(z)}{\Delta(z)}$$

where

$H(z) =$ the transfer function relating an output node to an input node

$\Delta(z) =$ the graph determinant

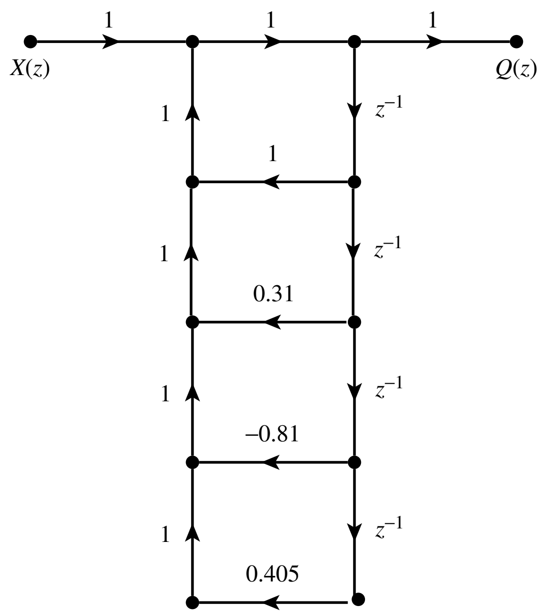
$P_k(z) =$ the gain of the kth path from input to output

$\Delta_k(z) =$ cofactor of the kth path.

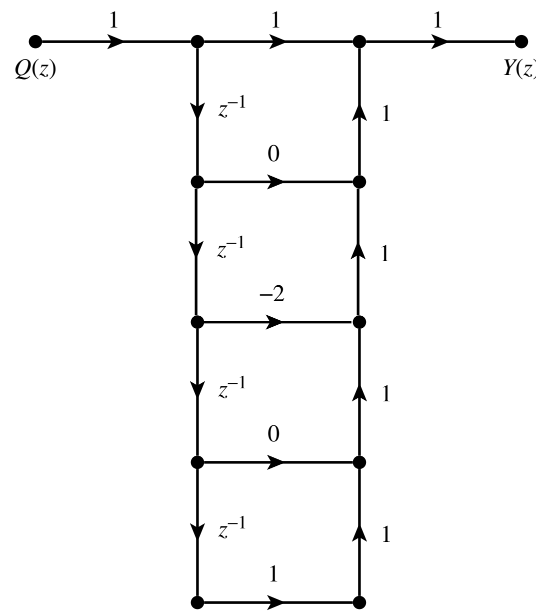
These terms will be defined as we find the transfer function $H(z) = Y(z)/X(z)$ for the SFG of Figure 3-12a on page 21, where there is a particular order required for the calculation of the three quantities $\Delta(z)$, $P_k(z)$, and $\Delta_k(z)$.

1. The first quantity to determine is the graph determinant $\Delta(z)$, where

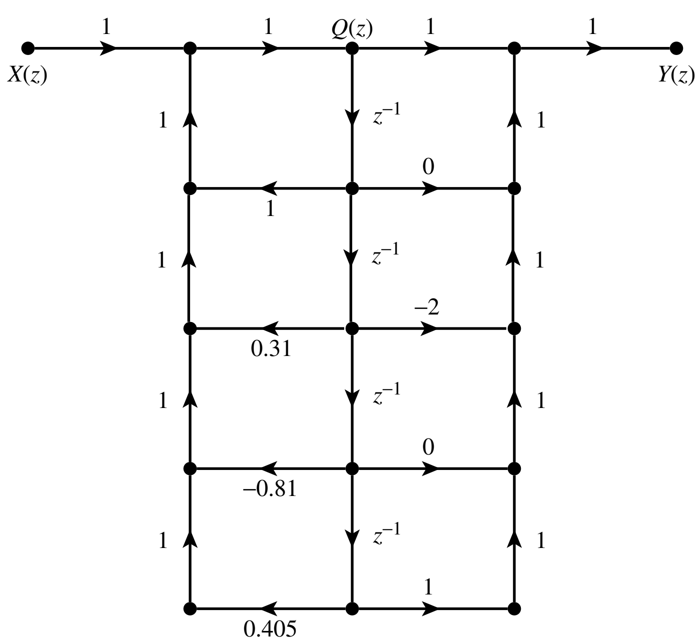
$$\begin{aligned}\Delta(z) = 1 &- \sum \text{loop gains} \\ &+ \sum \text{products of the gains of nontouching loops taken two at a time} \\ &- \sum \text{products of the gains of nontouching loops taken three at a time} \\ &+ \dots\end{aligned}$$



(a) Realization of poles of $H(z)$



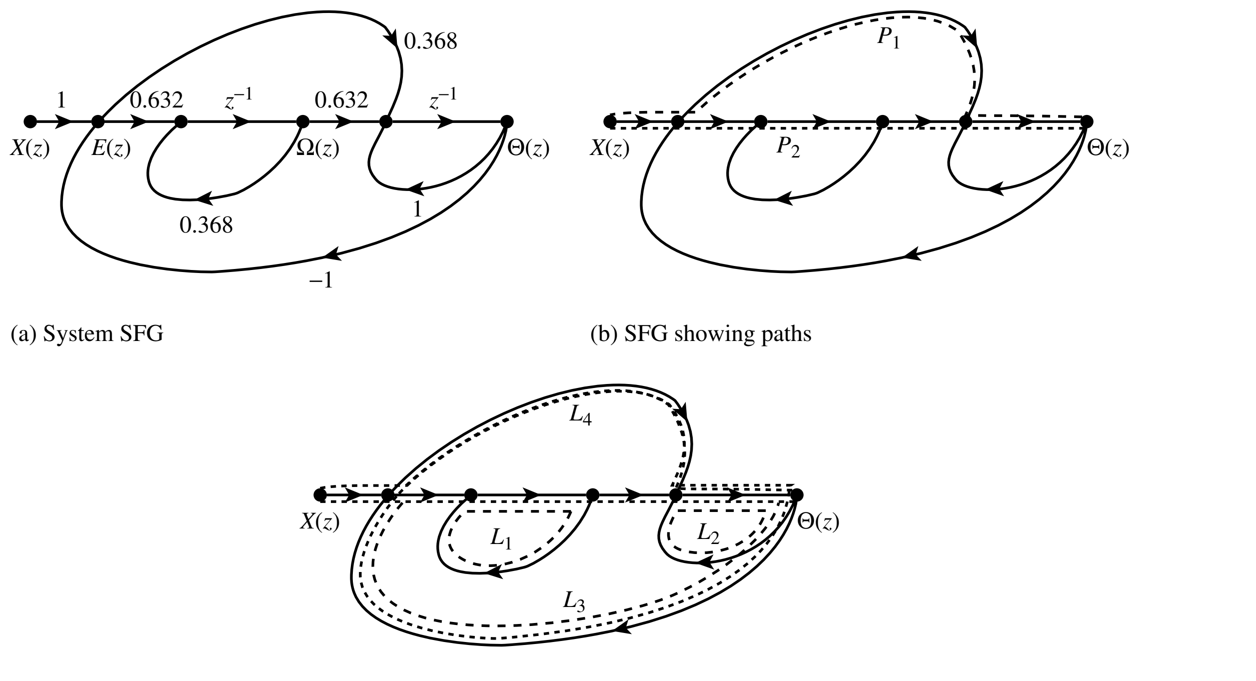
(b) Realization of zeros of $H(z)$



(c) Composite diagram for $H(z)$

Figure 3-11: System SFG for H(z)

where the loop gain L_j is simply the product of the gains around the jth loop. For the example of Figure 3-12c, the loop gains are $L_1 = 0.368z^{-1}$, $L_2 = 1z^{-1}$, $L_3 = (0.632)(z^{-1})(0.632)(z^{-1})(-1) = -(0.632)^2z^{-2}$, and $L_4 = (0.368)(z^{-1})(-1) = -0.368z^{-1}$. Loops L_1 and L_2 do not touch, nor do loops L_1 and L_4. In this



(a) System SFG

(b) SFG showing paths

(c) SFG showing loops

Figure 3-12: Signal flowgraph to illustrate Mason's Gain Rule

graph, we cannot find three loops that do not touch, so the calculation of $\Delta(z)$ terminates at this point, giving

$$\begin{aligned}
\Delta(z) &= 1 - \underbrace{(L_1 + L_2 + L_3 + L_4)}_{\text{Loop gains}} \\
&\quad + \underbrace{(L_1L_2 + L_1L_4)}_{\text{Gains of nontouching loops two at a time}} - 0 + 0 - \cdots \\
&= 1 - \left\{0.368z^{-1} + 1z^{-1} - (0.632)^2z^{-2} - 0.368z^{-1}\right\} \\
&\quad + \left\{(0.368z^{-1})(1z^{-1}) + (0.368z^{-1})(-0.368z^{-1})\right\} \\
&= 1 - z^{-1} + 0.632z^{-2}.
\end{aligned}$$

2. Next, we take up the path gains $P_k(z)$ and their cofactors $\Delta_k(z)$, where

$$\begin{aligned}
P_k(z) &= \text{gain of } k\text{th path from input to output} \\
&= \text{product of branch gains in } k\text{th path}
\end{aligned}$$

and

$$\Delta_k(z) = \text{cofactor of the } k\text{th path, formed by striking out from } \Delta(z) \text{ all terms associated with loops that are touched by the } k\text{th path.}$$

Consequently, for the SFG of Figure 3-12b where the paths are marked, we have

$$P_1(z) = (0.368)(z^{-1})(1) \quad \text{and} \quad \Delta_1(z) = 1 - 0.368z^{-1}$$

P_1 does not touch the α loop.

$$P_2(z) = (0.632)(z^{-1})(0.632)(z^{-1})(1) \quad \text{and} \quad \Delta_2(z) = 1.$$

P_2 touches all the loops and all terms except the 1 are stricken from $\Delta(z)$.

3. Finally, the transfer function is

$$\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= \frac{\sum_{k=1}^{2} P_k(z) \cdot \Delta_k(z)}{\Delta(z)} \\
&= \frac{0.368z^{-1}\left(1 - 0.368z^{-1}\right) + (0.632)^2 z^{-2}}{1 - z^{-1} + 0.632z^{-2}} \\
&= \frac{0.368z + 0.264}{z^2 - z + 0.632}.
\end{aligned}$$

Example 3-10: Practice with Mason's Gain Rule

For the digital-control system modeled in Figure 3-12a, use MGR to calculate the following transfer functions:

(a) $H_a(z) = \dfrac{\Omega(z)}{X(z)}$

(b) $H_b(z) = \dfrac{E(z)}{X(z)}$

Solution:

(a) Using the three steps outlined in the previous procedure gives the following results.

1. $\Delta(z) = 1 - z^{-1} + 0.632z^{-2}$. As before, the graph determinant $\Delta(z)$ is a function of the graph and is not affected by a designated input-output transfer function.
2. There is only one path from $X(z)$ to $\Omega(z)$, and it does not touch the β loop. Consequently, $P_1(z) = 0.632z^{-1}$ and $\Delta_1(z) = 1 - z^{-1}$.
3. Applying Mason's Gain Rule,

$$H_a(z) = \frac{\Omega(z)}{X(z)} = \frac{0.632z^{-1}(1 - z^{-1})}{1 - z^{-1} + 0.632z^{-2}}$$
$$= \frac{0.632(z-1)}{z^2 - z + 0.632}.$$

(b)

1. There is no change.
2. The one path does not touch loops L_1 and L_2, giving $P_1(z) = 1$ and

$$\Delta_1(z) = 1 - (0.368z^{-1} + 1z^{-1}) + (0.368z^{-1})(1z^{-1})$$
$$= 1 - 1.368z^{-1} + 0.368z^{-2}.$$

3. Once again with MGR,

$$H_b(z) = \frac{E(z)}{X(z)} = \frac{1 - 1.368z^{-1} + 0.368z^{-2}}{1 - z^{-1} + 0.632z^{-2}}$$
$$= \frac{z^2 - 1.368z + 0.368}{z^2 - z + 0.632}.$$

■

Comment:

A block-diagram model that is equivalent to the SFG of Figure 3-12a is given in Figure 3-13. Mason's Gain Rule may be used to calculate transfer functions, but the signs at the summations must be included when calculating loop and path gains.

3.5 The State-Space or First-Order Model

The fifth description of linear discrete-time systems treated in this chapter is called the state-space or first-order model. Used extensively in computer simulation of systems and theoretical investigations, this model represents difference equations of arbitrary order as a set of first-order difference equations. Assume an N^{th}-order, linear, constant coefficient difference equation

$$y(n) + a_1 y(n-1) + a_2 y(n-2)$$
$$+ \cdots + a_N y(n-N) = b_0 x(n)$$

or

$$\sum_{k=0}^{N} a_k y(n-k) = b_0 x(n)$$

where we have assumed that $a_0 = 1$, $y(n)$ is the output, $x(n)$ is the input, and there are N delays of the system output and no (zero) delays of the system input.[2] To represent this system by N first-order equations, we define N new variables $v_1(n), v_2(n), \ldots, v_N(n)$ as

$$v_1(n) = y(n-N),\ v_2(n) = y(n-N+1),$$
$$v_3(n) = y(n-N+2), \ldots,$$
$$v_r(n) = y(n-N+r-1), \ldots$$
$$v_N = y(n-1).$$

Then we can establish the following set of first-order difference equations, which are commonly called *state equations.*

State equations

$$v_1(n+1) = y(n-N+1) = v_2(n)$$
$$v_2(n+1) = y(n-N+1+1) = v_3(n)$$
$$\vdots$$
$$v_r(n+1) = y(n-N+r) = v_{r+1}(n)$$
$$\vdots$$
$$v_{N-1}(n+1) = y(n-1) = v_N(n)$$
$$v_N(n+1) = y(n) = -a_N v_1(n) - a_{N-1} v_2(n)$$
$$- \cdots - a_1 v_N(n) + b_0 x(n)$$

[2] The restriction of having no input delays, used here for simplicity, will be removed in Chapter 6.

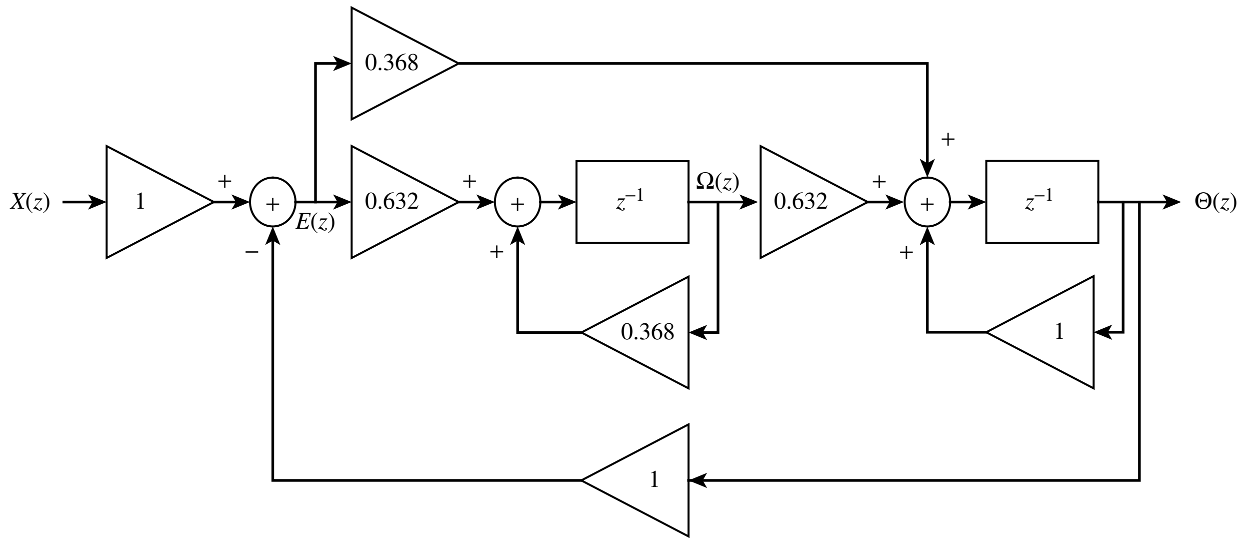


Figure 3-13: Block diagram equivalent to Figure 3-12a **SFG**

We next put this into matrix form, which is well suited for computation and the use of the tools of linear algebra. We call $v_1(n), v_2(n), \ldots, v_N(n)$ the *states* or *state variables* and define the state vector as

State vector

$$\mathbf{v}(n) = [v_1(n) v_2(n) \ldots v_N(n)]^T$$

where the superscript T stands for the matrix transpose operator. The matrix state equation is

Matrix state difference equation

$$\begin{bmatrix} v_1(n+1) \\ v_2(n+1) \\ \vdots \\ v_{N-1}(n+1) \\ v_N(n+1) \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_N & -a_{N-1} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \\ \vdots \\ v_{N-1}(n) \\ v_N(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} x(n)$$

that is,

$$\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}x(n)$$

where $\mathbf{A}$ is a matrix with N rows and N columns (an N by N or $N \times N$ matrix), $\mathbf{B}$ is an $N \times 1$ matrix (a so-called column vector)[3], $\mathbf{v}(n+1)$ and $\mathbf{v}(n)$ are $N \times 1$ matrices, and $x(n)$ is a 1×1 matrix or a scalar.

A definition for the state of a system is as follows:

The state of a system is a minimum set of quantities $v_1(n), v_2(n), \ldots, v_N(n)$, which if known at $n = n_0$, are determined (i.e., can be calculated) for $n > n_0$ by specifying the inputs to the system for $n \geq n_0$.

The *output* $y(n)$ of this single-input-single-output (SISO) system is related to the state vector $\mathbf{v}(n)$ and the single (scalar) input $x(n)$ by the *output equation*

$$y(n) = \mathbf{C}\mathbf{v}(n) + \mathbf{D}x(n)$$

where $\mathbf{C}$ is a $1 \times N$ matrix and $\mathbf{D}$ is a 1×1 (scalar) multiplier that may be written simply as d.

Comment:

Boldface letters, such as $\mathbf{v}(n)$, $\mathbf{A}$, $\mathbf{B}$, and so on, are used to denote matrices in this text. If a quantity that generally may be a matrix happens to be a scalar, regular typeface rather than boldface is used. This is the case, for example, for $x(n)$ in the two previous equations and for $y(n)$ in the output equation.

[3]With only one input, the matrix $\mathbf{B}$ could be written as $\mathbf{b}$, using the notation for a column vector.

When writing these equations by hand, boldface is not a convenient option, so an alternative is to write $\underset{\sim}{v}(n)$, $\underset{\sim}{A}$, $\underset{\sim}{B}$, and so forth and to omit the tilde when denoting a scalar quantity.

If you are rusty on the rules used for matrix operations, it is suggested that you consult Appendix B where the most common operations are reviewed.

It should be noted that the state-space formulation also can be used for systems having several inputs and several outputs, in which case we define the input and output vectors, respectively, as

$$\mathbf{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_R(n) \end{bmatrix} \quad \text{and} \quad \mathbf{y}(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \\ \vdots \\ y_P(n) \end{bmatrix}$$

and use vector-matrix notation to write the state and output equations as

$$\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n)$$
$$\mathbf{y}(n) = \mathbf{C}\mathbf{v}(n) + \mathbf{D}\mathbf{x}(n)$$

where $\mathbf{v}(n)$ is an $N \times 1$ matrix (a column vector with N elements), $\mathbf{x}(n)$ is an $R \times 1$ matrix (a column vector with R elements), $\mathbf{y}(n)$ is a $P \times 1$ matrix (a column vector with P elements), and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are matrices having dimensions $N \times N$, $N \times R$, $P \times N$, and $P \times R$, respectively.

Example 3-11: Finding the State and Output Equations

Find the state and output equations for the third-order system

$$y(n) - 0.25y(n-1) - 0.125y(n-2) + 0.5y(n-3) = 3x(n)$$

where $y(n)$ is the output and $x(n)$ is the input.

Solution: Defining $v_1(n) = y(n-3)$, $v_2(n) = y(n-2)$, and $v_3(n) = y(n-1)$, we have

$$v_1(n+1) = y(n-2) = v_2(n)$$
$$v_2(n+1) = y(n-1) = v_3(n)$$

and

$$v_3(n+1) = y(n) = 0.25v_3(n) + 0.125v_2(n) - 0.5v_1(n) + 3x(n).$$

In matrix form, the state and output equations are

$$\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & 0.125 & 0.25 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} x(n)$$
$$y(n) = \begin{bmatrix} -0.5 & 0.125 & 0.25 \end{bmatrix} \mathbf{v}(n) + 3x(n).$$

■

3.5.1 Iterative Solution of the State Equation in the Time Domain

Starting with

$$\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n)$$

we will assume that the $\mathbf{A}$ and $\mathbf{B}$ matrices are known. Also known are the initial condition (IC) vector $\mathbf{v}(0)$ with N components and the input vector $\mathbf{x}(n)$ for all integers $n \geq 0$, where we allow the possibility of R system inputs. The state vector $\mathbf{v}(n)$ has the same dimensions as the IC vector $\mathbf{v}(0)$, $N \times 1$, while $\mathbf{A}$ is a matrix with N rows and N columns—an $N \times N$ matrix—and $\mathbf{B}$ is $N \times R$.

First, for $n = 0$,

$$\mathbf{v}(1) = \mathbf{A}\mathbf{v}(0) + \mathbf{B}\mathbf{x}(0)$$

and for $n = 1$,

$$\begin{aligned} \mathbf{v}(2) &= \mathbf{A}\mathbf{v}(1) + \mathbf{B}\mathbf{x}(1) = \mathbf{A}\{\mathbf{A}\mathbf{v}(0) + \mathbf{B}\mathbf{x}(0)\} + \mathbf{B}\mathbf{x}(1) \\ &= \mathbf{A}^2\mathbf{v}(0) + \mathbf{A}\mathbf{B}\mathbf{x}(0) + \mathbf{B}\mathbf{x}(1). \end{aligned}$$

Continuing this iterative process leads to

$$\begin{aligned} \mathbf{v}(3) &= \mathbf{A}^3\mathbf{v}(0) + \mathbf{A}^2\mathbf{B}\mathbf{x}(0) + \mathbf{A}\mathbf{B}\mathbf{x}(1) + \mathbf{B}\mathbf{x}(2) \\ &\vdots \\ \mathbf{v}(n) &= \mathbf{A}^n\mathbf{v}(0) + \sum_{m=0}^{n-1} \mathbf{A}^{n-m-1}\mathbf{B}\mathbf{x}(m) \\ &= \mathbf{v}_{\mathbf{ZI}}(n) + \mathbf{v}_{\mathbf{ZS}}(n) \end{aligned}$$

where we see the tidy separation of the complete solution into the zero-input solution $\mathbf{v}_{\mathbf{ZI}}(n)$ caused by the initial-condition vector $\mathbf{v}(0)$ and the zero-state solution $\mathbf{v}_{\mathbf{ZS}}(n)$, that part of the solution due to the input vector $\mathbf{x}(m)$, $m = 0, 1, 2, \ldots, n-1$.[4]

[4] $\mathbf{v}_{\mathbf{ZI}}(n)$ also is known as the *initial-condition response*, and $\mathbf{v}_{\mathbf{ZS}}(n)$ is called the *forced response*.

Example 3-12: Iterative Solution of the State Equation

For the second-order system described by the state equation

$$\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(n)$$

with the initial condition $\mathbf{v}(0) = [2 \;\; 3]^T$ and the scalar exponential input sequence $x(n) = (0.5)^n u(n)$, use the iterative process to compute the state vector $\mathbf{v}(4)$.

Solution: For $n \geq 0$, the input sequence is $x(n) = \{1, 0.5, 0.25, 0.125, \ldots\}$.
For $n = 0$,

$$\mathbf{v}(1) = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1) = \begin{bmatrix} 3 \\ 0.5 \end{bmatrix}.$$

For $n = 1$,

$$\mathbf{v}(2) = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0.5) = \begin{bmatrix} 0.5 \\ -0.25 \end{bmatrix}.$$

For $n = 2$,

$$\mathbf{v}(3) = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.25 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0.25) = \begin{bmatrix} -0.250 \\ 0.125 \end{bmatrix}.$$

For $n = 3$,

$$\mathbf{v}(4) = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} -0.250 \\ 0.125 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0.125) = \begin{bmatrix} 0.1250 \\ 0.1875 \end{bmatrix}.$$

We also can compute the zero-input solution and the zero-state solution separately by using

$$\mathbf{v}(n) = \mathbf{v_{ZI}}(n) + \mathbf{v_{ZS}}(n) = \mathbf{A}^n \mathbf{v}(0) + \sum_{m=0}^{n-1} \mathbf{A}^{n-m-1} \mathbf{B} x(m).$$

$$\begin{aligned}
\mathbf{v_{ZI}}(4) &= \mathbf{A}^4 \mathbf{v}(0) = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 0.0625 & 0 \\ 0 & 0.0625 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.1250 \\ 0.1875 \end{bmatrix}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{v_{ZS}}(4) &= \sum_{m=0}^{3} \mathbf{A}^{3-m} \mathbf{B} x(m) \\
&= \mathbf{A}^3 \mathbf{B} x(0) + \mathbf{A}^2 \mathbf{B} x(1) + \mathbf{A}^1 \mathbf{B} x(2) + \mathbf{A}^0 \mathbf{B} x(3) \\
&= \begin{bmatrix} 0 & -0.25 \\ 0.0625 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1) \\
&\quad + \begin{bmatrix} -0.25 & 0 \\ 0 & -0.25 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0.5) \\
&\quad + \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0.25) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0.125) \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

■

3.5.2 Solution of the State Equation by z-Transforms

Given the matrix state equation for a linear time-invariant system

State equation

$$\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n)$$

with the known initial condition vector $\mathbf{v}(0)$. Taking the unilateral z-transform of this matrix equation and making use of the time-shift property that was derived in Chapter 2 for unilateral transforms gives

Transformed state equation[a]

$$z\mathbf{V}(z) - z\mathbf{v}(0) = \mathbf{A}\mathbf{V}(z) + \mathbf{B}\mathbf{X}(z).$$

[a] Notice that here we use capital boldface letters with the argument z, as in $\mathbf{V}(z) = \mathbb{Z}[\mathbf{v}(n)]$, to denote the z-transform of a matrix quantity.

Notice that $\mathbf{v}(0)$ is an $N \times 1$ matrix of constants—the initial conditions.

Moving the unknown $\mathbf{V}(z)$ to the left side and the known $\mathbf{v}(0)$ to the right side, we have

$$z\mathbf{V}(z) - \mathbf{A}\mathbf{V}(z) = z\mathbf{v}(0) + \mathbf{B}\mathbf{X}(z).$$

By writing the vector $z\mathbf{V}(z)$ as $z\mathbf{I}\mathbf{V}(z)$, where $\mathbf{I}$ is the $N \times N$ identity matrix and post factoring $\mathbf{V}(z)$, we obtain

$$z\mathbf{I}\mathbf{V}(z) - \mathbf{A}\mathbf{V}(z) = (z\mathbf{I} - \mathbf{A})\mathbf{V}(z) = z\mathbf{v}(0) + \mathbf{B}\mathbf{X}(z).$$

Premultiplying both sides of this equation by $(z\mathbf{I} - \mathbf{A})^{-1}$, the inverse of the matrix $z\mathbf{I} - \mathbf{A}$, gives[5]

z-transform of $\mathbf{v}(n)$

$$\mathbf{V}(z) = z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{v}(0) + (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{X}(z).$$

Defining $\mathbf{\Phi}(z) = z(z\mathbf{I} - \mathbf{A})^{-1}$ gives the notationally neater form

$$\begin{aligned}
\mathbf{V}(z) &= \mathbf{\Phi}(z)\mathbf{v}(0) + \frac{\mathbf{\Phi}(z)}{z}\mathbf{B}\mathbf{X}(z) \\
&= \mathbf{V_{ZI}}(z) + \mathbf{V_{ZS}}(z).
\end{aligned}$$

[5] It can be shown that the matrix $z\mathbf{I} - \mathbf{A}$ is nonsingular, and thus, its inverse $[z\mathbf{I} - \mathbf{A}]^{-1}$ exists. So $[z\mathbf{I} - \mathbf{A}]^{-1} \cdot [z\mathbf{I} - \mathbf{A}] = \mathbf{I}$, which is the identity matrix.

where we recognize the first term as the z-transform of the zero-input response and the second term as the z-transform of the zero-state response.

Taking the inverse z-transform of the transformed state vector $\mathbf{V}(z)$ gives

$$\begin{aligned}\mathbb{Z}^{-1}[\mathbf{V}(z)] &= \mathbf{v}(n)\\ &= \mathbb{Z}^{-1}[\boldsymbol{\Phi}(z)\mathbf{v}(0)] + \mathbb{Z}^{-1}\left[\frac{\boldsymbol{\Phi}(z)\mathbf{B}\mathbf{X}(z)}{z}\right]\end{aligned}$$

or since $\mathbf{v}(0)$ is a matrix of constants,

$$\begin{aligned}\mathbf{v}(n) &= \boldsymbol{\phi}(n)\mathbf{v}(0) + \sum_{m=1}^{n}\boldsymbol{\phi}(n-m)\mathbf{B}\mathbf{x}(m-1)\\ &= \boldsymbol{\phi}(n)\mathbf{v}(0) + \sum_{m=0}^{n-1}\boldsymbol{\phi}(n-m-1)\mathbf{B}\mathbf{x}(m)\\ &= \mathbf{v}_{\mathbf{ZI}}(n) + \mathbf{v}_{\mathbf{ZS}}(n),\end{aligned}$$

where $\boldsymbol{\phi}(n)$ is the inverse z-transform of the matrix $\boldsymbol{\Phi}(z)$; i.e., the ijth element of $\boldsymbol{\phi}(n)$ is the inverse z-transform of the ijth element of $\boldsymbol{\Phi}(z)$. *The matrix $\boldsymbol{\phi}(n)$ is called the state transition matrix*; it determines the evolution of the system's zero-input response from the initial state value $\mathbf{v}(0)$ to $\mathbf{v}(n)$ and it also affects the zero-state response that appears as the second term on the right side of the equation for $\mathbf{v}(n)$. The zero-state response $\mathbf{v}_{ZS}(n)$ is given by the *matrix convolution sum*. As with convolution involving scalar functions, numerical answers are generally obtained more directly by using z-transforms or numerical procedures.

Example 3-13: z-Transform Solution of the State Equation

An LTI discrete-time system is described by the state equation of Example 3-12 as

$$\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix}\mathbf{v}(n) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}x(n).$$

(a) Find the state transition matrix $\boldsymbol{\phi}(n)$.

(b) Find an analytical solution for the state vector $\mathbf{v}(n)$ with $\mathbf{v}(0) = [2\ 3]^T$ and $x(n) = 0.5^n u(n)$.

(c) Using the numerical initial conditions and input of part b, determine the numerical value of $\mathbf{v}(4)$ and compare with the result of Example 3-12.

Solution:

(a) Following the steps just given, we have

$$z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 0.25 & z \end{bmatrix}.$$

The inverse of a square matrix $\mathbf{M}$ can be written as the adjoint matrix, adj($\mathbf{M}$), divided by the determinant of $\mathbf{M}$ (see Appendix B), or

$$\mathbf{M}^{-1} = \frac{\text{adj}(\mathbf{M})}{|\mathbf{M}|}.$$

Thus, for the inverse of $(z\mathbf{I} - \mathbf{A})$, we have

$$(z\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adj}(z\mathbf{I} - \mathbf{A})}{|z\mathbf{I} - \mathbf{A}|} = \frac{1}{z^2 + 0.25}\begin{bmatrix} z & -0.25 \\ 1 & z \end{bmatrix}$$

and

$$\boldsymbol{\Phi}(z) = z(z\mathbf{I} - \mathbf{A})^{-1} = z\begin{bmatrix} \dfrac{z}{z^2 + 0.25} & \dfrac{-0.25}{z^2 + 0.25} \\ \dfrac{1}{z^2 + 0.25} & \dfrac{z}{z^2 + 0.25} \end{bmatrix}.$$

In the usual way, we expand each entry of $\boldsymbol{\Phi}(z)/z$ in partial fractions and then multiply by z before evaluating the inverse transforms with the aid of Table 2.1. The result is

$$\boldsymbol{\Phi}(n) = \begin{bmatrix} (0.5)^n\cos(n\pi/2) & -0.25\cos(n\pi/2 - \pi/2) \\ \cos(n\pi/2 - \pi/2) & (0.5)^n\cos(n\pi/2) \end{bmatrix}.$$

CROSS CHECK A good check here is to evaluate $\boldsymbol{\phi}(0)$, which always should be equal to the identity matrix. By inspection, we see that this is the case, i.e., $\boldsymbol{\phi}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) To find the complete response, we need to evaluate

$$\mathbf{V}(z) = \left[\boldsymbol{\Phi}(z)\mathbf{v}(0) + \frac{\boldsymbol{\Phi}(z)\mathbf{B}\mathbf{X}(z)}{z}\right]$$

and then find the inverse transforms. Carrying out the indicated operations with $\boldsymbol{\Phi}(z)$ as found previously and

$$\mathbf{v}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad X(z) = \frac{z}{z - 0.5}$$

yields, after some algebra,

$$V(z) = \begin{bmatrix} \dfrac{2z^3 - 1.75z^2 + 0.125z}{z^3 - 0.5z^2 + 0.25z - 0.125} \\ \dfrac{3z^3 + 1.5z^2 - z}{z^3 - 0.5z^2 + 0.25z - 0.125} \end{bmatrix}.$$

As usual, we then expand $V(z)/z$ in partial fractions and use the m-function `residue` to find the PFE coefficients. After doing this and multiplying by z to obtain the z-transform forms of Table 2.1, we have

$$V_1(z) = \frac{-0.4z}{z-0.5} + \frac{1.320ze^{j0.43}}{z-0.5e^{j\pi/2}} + \frac{1.320ze^{-j0.43}}{z-0.5e^{-j\pi/2}}$$
$$V_2(z) = \frac{z}{z-0.5} + \frac{2.693ze^{-j1.19}}{z-0.5e^{j\pi/2}} + \frac{2.693ze^{j1.19}}{z-0.5e^{-j\pi/2}}.$$

Using Table 2.1 and combining the second and third terms in each PFE gives

$$v_1(n) = [-0.4(0.5)^n + 2.640(0.5)^n \cos(n\pi/2 + 0.43)]u(n)$$
$$v_2(n) = [(0.5)^n + 5.386(0.5)^n \cos(n\pi/2 - 1.19)]u(n).$$

(c) Evaluating these expressions for $v_1(n)$ and $v_2(n)$ with $n = 4$ yields

$$v_1(4) = 0.125 \quad \text{and} \quad v_2(4) = 0.1875$$

as found in Example 3-12. ■

Comment:

This is a lot of work, and there are several opportunities to make algebraic errors. If it is numerical answers we are after instead of analytical expressions as in the previous example, using m-files generally is a better approach.

3.5.3 State Equations from System Diagrams

Thus far, we have developed state equations from a system's DE, but it also is possible to write state and output equations from a system diagram. We assume that the system diagram for an N^{th}-order system is drawn so that the N delay element building blocks appear as shown in Figure 3-6a. This is sometimes called *the all unit delay form* and has been the case for all system diagrams we have seen thus far. In this situation, the following procedure can be applied.

Step 1: Label the outputs of all delays as state variables, the delays can be numbered in any order, but there must be N states.

Step 2: At the input to the delay whose output is defined as $v_j(n)$ write a state equation of the form

$$v_j(n+1) = \text{ weighted sum of}$$
$$v_1(n), v_2(n), \ldots, v_N(n) \text{ and } x(n).$$

Repeat this for each delay element.

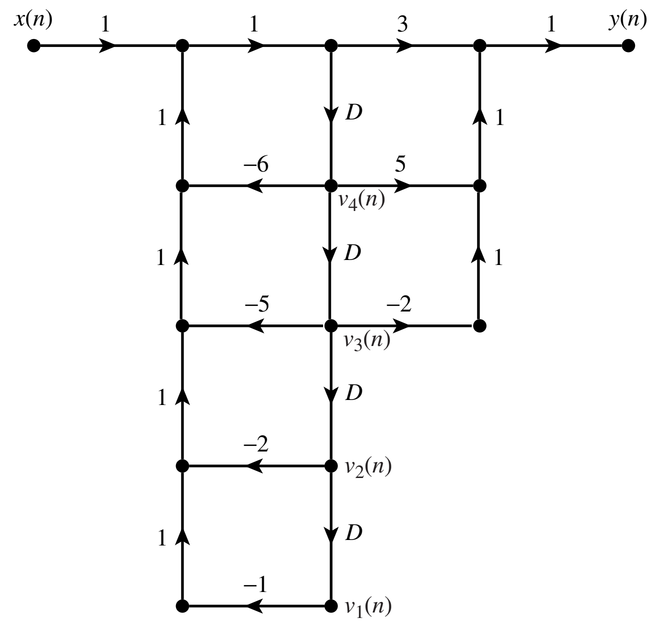


Figure 3-14: Signal flowgraph for fourth-order system

Step 3: Obtain an equation for the output (or outputs, if there are more than one) $\mathbf{y}(n)$. This should be reduced so that $\mathbf{y}(n)$ is expressed as a linear combination of the N states as

$$v_1(n), v_2(n), \ldots, v_N(n) \text{ and the input } x(n)$$

Step 4: Put the state and output equations in matrix form if desired.

The following example illustrates this procedure.

Example 3-14: State Equations from a Signal Flowgraph

Write state and output equations in matrix form for the system shown in Figure 3-14.

Solution: We follow the procedure just outlined.

Step 1: The outputs of the delays have been labeled in Figure 3-14.

Step 2: At the inputs of the delays we have

$$v_1(n+1) = v_2(n)$$
$$v_2(n+1) = v_3(n)$$
$$v_3(n+1) = v_4(n)$$
$$v_4(n+1) = -v_1(n) - 2v_2(n) - 5v_3(n) - 6v_4(n) + x(n).$$

Step 3: The output $y(n)$ is given by

$$\begin{aligned} y(n) &= 3v_4(n+1) - 2v_3(n) + 5v_4(n) \\ &= 3\left[-v_1(n) - 2v_2(n) - 5v_3(n) - 6v_4(n) + x(n)\right] \\ &\quad - 2v_3(n) + 5v_4(n) \\ &= -3v_1(n) - 6v_2(n) - 15v_3(n) - 18v_4(n) \\ &\quad + 3x(n) - 2v_3(n) + 5v_4(n) \\ &= -3v_1(n) - 6v_2(n) - 17v_3(n) - 13v_4(n) + 3x(n). \end{aligned}$$

Step 4: In matrix form, the state and output equations are

$$\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -5 & -6 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x(n)$$

$$y(n) = \begin{bmatrix} -3 & -6 & -17 & -13 \end{bmatrix} \mathbf{v}(n) + 3x(n). \quad \blacksquare$$

3.6 Stability

As we observed in Chapter 2, there are several ways of defining linear system stability. Let's review the three ways we considered.

Bounded Input, Bounded Output (BIBO) Stability: A system is stable if and only if the system output is bounded (does not approach $\pm\infty$) for all bounded inputs.

Zero-input Response: A causal LTI system is stable if and only if the zero input (or initial condition) response approaches zero as $n \to \infty$.

Absolute Summability of Unit Impulse Response: A system is stable if and only if the unit impulse response is absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} \left|h(n)\right| < \infty$$

The most useful test for stability of LTI discrete-time systems is the zero-input response test because it can be related directly to the system pole or characteristic root locations. The BIBO condition is least useful as a test because it is necessary to guarantee bounded outputs *for all* bounded inputs. Absolute summability of the unit impulse response is useful in some cases, as we will see later in this section.

We will restrict our attention to the case where we have a causal, LTI discrete-time system in IIR form. We can view FIR systems as the special case when there are no output delays. Later, we also will generalize the results to include noncausal systems. The systems will be modeled in one of the four forms considered previously.

Linear difference equation

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k)$$

Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{L} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Unit impulse response

$$h(n) = \sum_{k=1}^{N} C_k (d_k)^n u(n) + \sum_{k=0}^{L-N} A_k \delta(n-k)$$

State difference equations

$$\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n)$$

We assume for simplicity that the system poles (the roots of the characteristic equation) are distinct. This restriction can be removed, however, without altering the results obtained. It is important to keep in mind that stability is a characteristic of a system, *not* of its input. A stable system can be driven to infinity by an unbounded input, but it is the opposite situation we generally try to avoid: a system whose response to a bounded input approaches $\pm\infty$ as $n \to \infty$.

3.6.1 Stability of Linear Difference Equation Models

For a system modeled by the N^{th}-order recursive difference equation

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k)$$

the zero-input response is given by

$$y_{ZI}(n) = \sum_{k=1}^{N} C_k (d_k)^n$$

when the characteristic equation has distinct roots.

A *causal* LTI system is said to be stable if and only if the zero-input response decays to zero as $n \to \infty$. This happens if and only if all of the d_k's have magnitudes *less than* 1.

A nonrecursive system is described by

$$y(n) = \sum_{k=0}^{L} b_k x(n-k)$$

and is always stable, because its zero-input response is zero for all n.

3.6.2 Stability of Unit Impulse Response Models

Another definition of system stability is expressed in terms of the system's input and output sequences—that is, in a stable system, all bounded input sequences produce bounded output sequences. This definition leads to the easier to test requirement that a stable system must have an absolutely summable unit impulse response, that is,

Stability definition

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Since the unit impulse response of a nonrecursive or FIR system is given by

$$\begin{aligned} h(n) &= \sum_{k=0}^{L} b_k \delta(n-k) \\ &= \sum_{k=0}^{L} h(k)\delta(n-k) \end{aligned}$$

we see that FIR systems are always stable because

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{k=0}^{L} |h(k)| \\ &= \sum_{k=0}^{L} |b_k| = S < \infty \end{aligned}$$

which is a finite sum.

From Section 3.2, the unit impulse response for a causal nonrecursive system is given by

$$h(n) = \sum_{k=1}^{N} C_k (d_k)^n + \sum_{k=0}^{L-N} A_k \delta(n-k).$$

It is the case that the unit impulse response is absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

if (and only if) the magnitudes of all of the characteristic roots are less than 1. Thus for a stable causal recursive system we again have

$$|d_k| < 1, \quad k = 1, 2, \ldots, N.$$

The details are available on the FPALS website.

Example 3-15: Unit Impulse Response and Stability

Find the unit impulse response $h(n)$ for the two following systems. In each case, determine if the system is stable.

(a) A comb filter that can be used in radar signal processing applications described by the difference equation $y(n) = x(n) - x(n-8)$.

(b) A causal, first-order highpass filter modeled by $y(n) + 0.9y(n-1) = 2x(n)$.

Solution:

(a) The comb filter is an FIR (nonrecursive) system where $h(n) = b_n$ and $h(n) = \delta(n) - \delta(n-8)$. This FIR system is stable, since

$$\sum_{n=-\infty}^{\infty} |h(n)| = 2.$$

(b) For the recursive filter $a_1 = 0.9$ and $b_0 = 2$, and by comparison with the expression derived earlier for a first-order system, $h(n) = 2(-0.9)^n u(n)$. Here, using the infinite geometric sum formula

$$\sum_{n=-\infty}^{\infty} |h(n)| = 2\sum_{n=0}^{\infty} |(-0.9)^n| = \frac{2}{1-0.9} = 20.$$

Hence, it is a stable system. From another point of view, the characteristic root is $d_1 = -0.9$, its magnitude is less than 1, and the causal system is stable.

A plot (using `filter`) of the unit impulse response $h(n) = 2(-0.9)^n u(n)$ is given in Figure 3-15, where it is evident that this is an absolutely summable response, that is, a stable system.

■

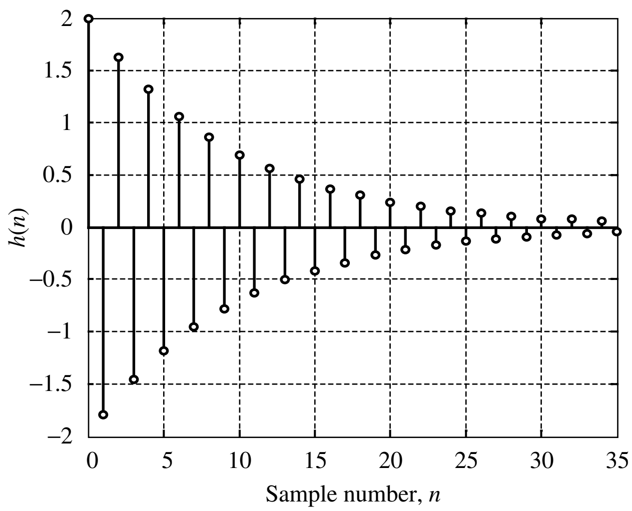


Figure 3-15: Unit impulse response of a stable IIR system

3.6.3 Stability of Transfer Function Models

Stability also may be deduced from transfer functions. Consider a causal LTI system with distinct characteristic roots d_k as described by the unit impulse response

Unit impulse response

$$h(n) = A_0\delta(n) + \sum_{k=1}^{N} C_k d_k^n u(n),\ N = L$$

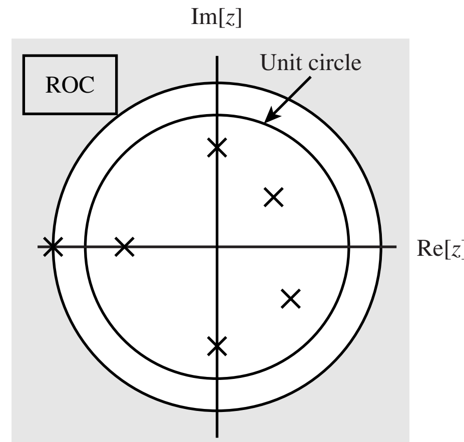
with the corresponding transfer function

Transfer function

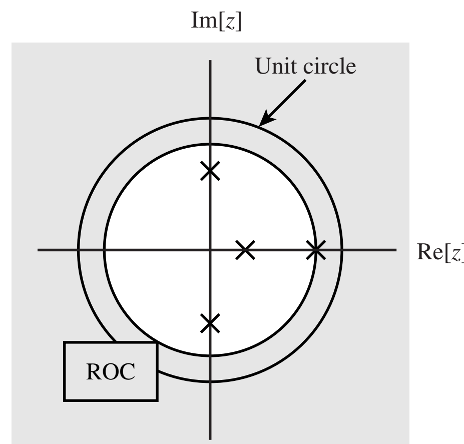
$$H(z) = A_0 + \sum_{k=1}^{N} \frac{C_k z}{z - d_k},\ |z| > |d_k|.$$

The region of convergence is the area outside the circle centered at the origin and passing through the pole d_k having the largest magnitude, as shown in Figure 3-16a. If the system is stable, then we deduce from the equations for $h(n)$ and $H(z)$ that the magnitudes of all of the system poles must be less than 1. For this situation, we have Figure 3-16b, where we see that the region of convergence includes the unit circle.[6]

[6]Although we've restricted consideration here to the predominant case of nonrepeated poles, the result that the ROC must contain the unit circle for stability applies as well to the multiple-pole case.



(a) ROC for a causal system



(b) ROC for a stable causal system

Figure 3-16: Stability of casual systems and ROC

3.6.4 Stability of State Equation Models

As noted earlier, one definition of stability for a causal LTI system is the following.

The zero-input response of a stable causal system approaches zero as $n \to \infty$ for all initial conditions.

We now investigate this from the point of view of the state equation. First, the zero-input response is given by

$$\mathbf{v}_{\mathbf{ZI}}(n) = \mathbf{A}^n \mathbf{v}(0)$$

and in order that $\mathbf{v}(n) \to \mathbf{0}$ for arbitrary $\mathbf{v}(0)$, we see that $\mathbf{A}^n$ must approach the null matrix $\mathbf{0}$ as $n \to \infty$, that is, all of the elements of $\mathbf{A}^n$ must approach zero as n approaches infinity.

Example 3-16: Evaluation of Stability

Given the state equation $\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n)$ with different coefficient (A) matrices, as shown, use the concept of raising $\mathbf{A}$ to the nth power to estimate the stability of the following:

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 0.632 \\ 0 & 0.368 \end{bmatrix}$

(d) $\mathbf{A} = \begin{bmatrix} 0.632 & 0.632 \\ 0.632 & 0.368 \end{bmatrix}$

Solution: Using $n = 10$ (any fairly large integer will do), we find the following by evaluating the matrix power $\mathbf{A}^n$ by writing and executing an m-file.

(a) $\mathbf{A}^{10} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, stable.
In fact, $\mathbf{A}^n = \mathbf{0}$ for $2 \leq n$.

(b) $\mathbf{A}^{10} = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}$, unstable

(c) $\mathbf{A}^9 = \begin{bmatrix} 1.0 & 0.9999 \\ 0 & 0.0001 \end{bmatrix}$,

$\mathbf{A}^{10} = \mathbf{A}^{11} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ unstable

(d) $\mathbf{A}^{10} = \begin{bmatrix} 2.3455 & 1.9062 \\ 1.9062 & 1.5492 \end{bmatrix}$, unstable

■

3.6.5 Characteristic Equation and Characteristic Roots (Eigenvalues)

In Section 3.5.2, we found that the matrix $z\mathbf{I} - \mathbf{A}$ has the characteristic polynomial.

Characteristic polynomial

$$|z\mathbf{I} - \mathbf{A}| = z^N + \alpha_{N-1} z^{N-1} + \cdots + \alpha_1 z + \alpha_0$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ are constants that depend on system parameters. This polynomial plays a vital role in the dynamic behavior of the system. When equated to zero, this characteristic polynomial becomes the system's characteristic equation (CE)

Characteristic equation

$$z^N + \alpha_{N-1} z^{N-1} + \cdots + \alpha_1 z + \alpha_0 = 0.$$

This CE can be written in factored form as

Factored CE

$$(z - d_1)(z - d_2) \cdots (z - d_N) = 0$$

where $z_1 = d_1, z_2 = d_2 \ldots z_N = d_N$ are the *characteristic roots* or *eigenvalues* of the system. These roots determine the essential features of the zero-state (initial condition) behavior of the system, and for a stable causal system of all these roots must have magnitudes less than 1. These characteristic roots also create the terms in the partial fraction expansion of the matrix $\mathbf{\Phi}(z)$ whose inverse z-transform is the *transition matrix* $\boldsymbol{\phi}(n)$. The eigenvalues, usually denoted as λ_k, $k = 1, 2, \ldots, N$, and the characteristic roots are the roots of the characteristic equation $|\lambda\mathbf{I} - \mathbf{A}| = 0$. Notice that the roots of system's characteristic equation, the eigenvalues, and the system poles are all the same thing; these terms are simply aliases for one another!

Example 3-17: Computer Control of an Analog System

A continuous-time linear system is supplied with a piecewise-constant input, $x(t) = x(nT)$, $nT \leq t < nT + T$, that comes from a digital controller. The system is described at the sampling instants by the matrix difference equation and by the output equation

$$\mathbf{v}(n+1) = \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0.1667 \\ 0.5000 \\ 1 \end{bmatrix} x(n)$$

and

$$y(n) = [1 \quad 0 \quad 0]\, \mathbf{v}(n).$$

Find the system's characteristic equation. What are the eigenvalues? Is the system stable? Reinforce your answer to the stability issue by using the m-function `dlsim` to plot a few zero-input responses.

Solution: The characteristic polynomial is formed from $|z\mathbf{I} - \mathbf{A}|$ or $|\lambda\mathbf{I} - \mathbf{A}|$ by expanding the following determinant about the first row as

$$\begin{vmatrix} z-1 & -1 & -0.5 \\ 0 & z-1 & -1 \\ 0 & 0 & z-1 \end{vmatrix} = (z-1)(z-1)(z-1) + 1(0) - 0.5(0).$$

This gives the characteristic equation

$$(z-1)(z-1)(z-1) = 0 \text{ or } (z-1)^3 = 0$$
$$z^3 - 3z^2 + 3z - 1 = 0.$$

We find the characteristic roots or eigenvalues to be $z_{1,2,3} = 1$. An m-file that omits the plotting statements is given here. The system is, consequently, unstable, as can be seen in Figure 3-17, where the zero-input $\mathbf{v}(0) = [111]^T$ response, apparently increasing without bound, is plotted.

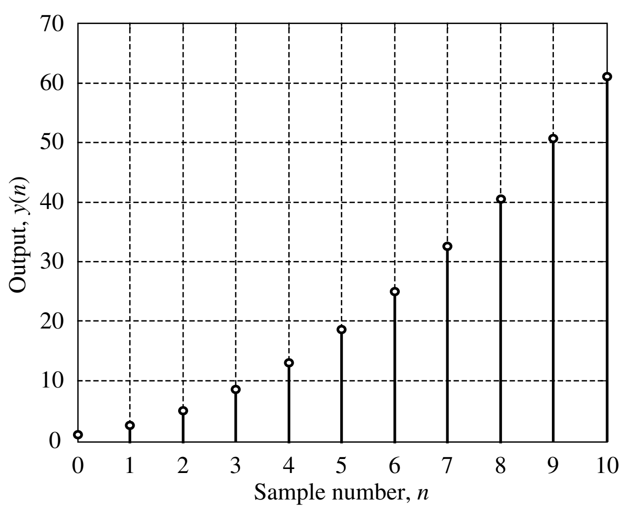


Figure 3-17: Zero-input response of system

m-file

```
%F3_17 Computer control of an analog system
% F3_17 Zero-input response of system
n=0:1:10;
A=[1,1,0.5;0,1,1;0,0,1];
B=[0.1667;0.500;1];
C=[1,0,0];
D=0;
v0=[1;1;1];
X=[0*n]';
[y,v]=dlsim(A,B,C,D,X,v0);
% ... plotting statements
```

3.6.6 Stability and Regions of Convergence Revisited

Let us now generalize some of our previous results to the case of anticausal and noncausal systems. Even though such systems are not actually implemented, they are sometimes used to facilitate digital-filter design approaches. In addition, considering these systems allows us to develop a unified framework for viewing system stability and its connection to regions of convergence.

Anticausal systems:

A recursive anticausal LTI system is described by the unit impulse response[7]

$$h(n) = \sum_{k=1}^{N} C_k d_k^n u(-n-1)$$

and the corresponding transfer function

Transfer function

$$H(z) = -\sum_{k=1}^{N} \frac{C_k z}{z - d_k}, \; |z| < |d_k|$$

where the region of convergence is the area inside of the circle centered at the origin and passing through the pole d_k having the smallest magnitude. If the system is stable, that is, if

[7]This formulation ignores the possible presence of any shifted impulses which do not affect stability.

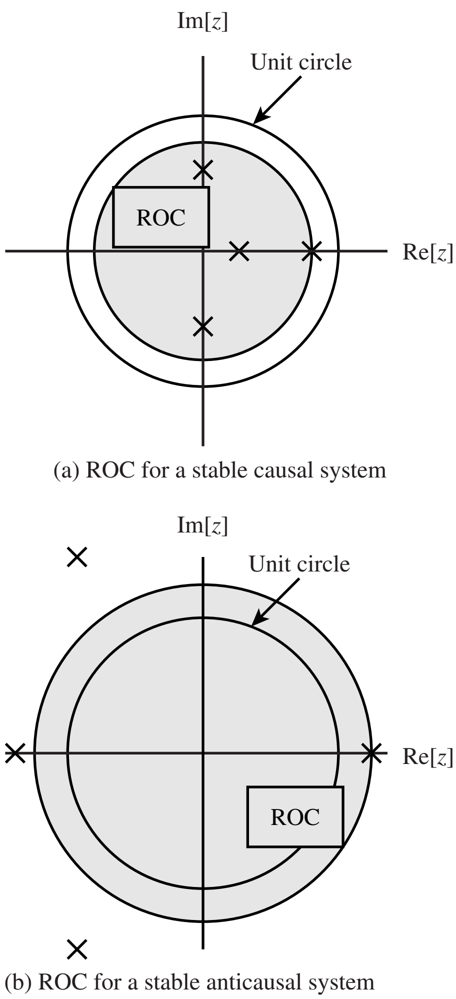


Figure 3-18: ROCs for stable causal and anticausal systems

Summable impulse response

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

then we see from the equations for $h(n)$ and $H(z)$ that the magnitudes of all of the system poles must be greater than 1. Keep in mind that the step sequence $u(-n-1)$ is 1 for $n \leq -1$ and 0 for $n \geq 0$, so the pole values in the expression for $h(n)$ are being raised to negative powers. For this situation, we have Figure 3-18b, where we see that the region of convergence is inside the circle centered at the origin and passing through the smallest pole d_k of the system transfer function $H(z)$ and that the ROC includes the unit circle.

Coupled with the previous results from Section 3.6.1, we now have the statement:

Stable causal systems have all of their poles inside the unit circle, whereas stable anticausal systems have all of their poles outside the unit circle. For stable systems, consequently, the unit circle is included in the region of convergence.

Noncausal systems:

Noncausal systems, which have both a causal and an anticausal part, are used frequently in an intermediate step in the design of signal-processing systems. The poles of the "causal" part of the system must lie inside the unit circle, while those of the "anticausal" part must lie outside for the overall noncausal system to be stable.

Example 3-18: Stability and the Region of Convergence

Given the system transfer function

$$H(z) = \frac{9z^2 - 12z}{z^2 - 2.5z - 1.5}$$

determine stability and the form of $h(n)$ for the following regions of convergence.

(a) $|z| < 0.5$

(b) $0.5 < |z| < 3$

(c) $|z| > 3$

Solution: The system poles are $z_1 = -0.5$ and $z_2 = +3$.

(a) From the given ROC, we conclude that the system is anticausal. Since the unit circle is not included in the ROC, the system is unstable with

$$h(n) = C_1(-0.50)^n u(-n-1) + C_2(3)^n u(-n-1).$$

(b) The annular form of the ROC indicates that the system is noncausal. The unit circle is included in the ROC, so the system is stable with

$$h(n) = C_3(-0.50)^n u(n) + C_4(3)^n u(-n-1).$$

(c) The form of the ROC indicates that the system is causal, but it is unstable because the ROC does not include the unit circle. The unit impulse response is

$$h(n) = C_5(-0.50)^n u(n) + C_6(3)^n u(n).$$

■

Definitions, Techniques, and Connections

Bilateral z-transform	$\mathbb{Z}[f(n)] = F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n}$
Inverse z-transform	$\mathbb{Z}^{-1}[F(z)] = \frac{1}{2\pi j}\oint_C F(z)z^{n-1}dz$
Region of convergence	ROC; region of z for which $F(z)$ exists.
Notation	$f(n) \Leftrightarrow F(z)$
Pairs	Table 2.1
Unilateral transform	$\mathbb{Z}[f(n)u(n)] = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$
Properties of bilateral transforms	Table 2.2

Transfer function

$$H(z) = \frac{Y(z)}{X(z)}, \text{ICs} = 0$$

1. $H(z) = \dfrac{z\text{-transform of the zero-state output sequence } y(n)}{z\text{-transform of the input sequence } x(n)}$

2. $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\sum_{k=0}^{L} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$

given the DE

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k).$$

3. $H(z) = \mathbb{Z}[h(n)]$

4. Mason's Gain Rule (MGR)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=1}^{R} P_k(z)\cdot\Delta_k(z)}{\Delta(z)}$$

Zeros and poles

$$H(z) = \frac{b_0 z^{N-L}(z-n_1)(z-n_2)\cdots(z-n_L)}{(z-d_1)(z-d_2)\cdots(z-d_N)}$$

Stability from $\boldsymbol{H(z)}$

1. Causal system: magnitudes of all poles are less than 1.
2. Anticausal system: magnitudes of all poles are greater than 1.
3. General: ROC includes the unit circle.

Inverse transforms

1. From the definition: $C_k\delta(n-k) \Leftrightarrow C_k z^{-k}$
2. From long division: $P(z) = N(z)/D(z)$, where
   Causal sequence
   $P(z) = \alpha_0 z^0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_{17} z^{-17} + \cdots$
   Anticausal sequence
   $$P(z) = \beta_1 z^1 + \beta_2 z^2 + \cdots + \beta_{17} z^{17} + \cdots$$
3. Partial fraction expansion, distinct poles, degrees of numerator and denominator of $F(z)$ the same
   $$F(z) = K\frac{B(z)}{(z-d_1)(z-d_2)\cdots(z-d_R)} = C_0 + \sum_{k=1}^{R}\frac{C_k z}{z-d_k}$$
   where the partial fraction constants are determined from the expansion
   $$\frac{F(z)}{z} = K\frac{B(z)}{z(z-d_1)(z-d_2)\cdots(z-d_R)} = \frac{C_0}{z} + \frac{C_1}{z-d_1} + \frac{C_2}{z-d_2} + \cdots + \frac{C_N}{z-d_R}$$
   and knowing the ROC, the sequence $f(n)$ may be found using Table 2.1.

Convolution

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m) \Leftrightarrow H(z)\cdot X(z)$$

Sinusoidal response and frequency response

For $x(n) = A\cos(n\theta+\beta)u(n)$ and $H(e^{j\theta}) = \left|H(e^{j\theta})\right| e^{j\angle H(e^{j\theta})}$,

$$y_{ss}(n) = A\left|H(e^{j\theta})\right|\cos\left[n\theta + \beta + \angle \mathrm{H}(e^{j\theta})\right].$$

State Space or First-Order Model for The DE

General definition The state of a system is a minimum set of quantities $v_1(n)$, $v_2(n)$, $\ldots$, $v_N(n)$, which if known at $n = n_0$ are determined for $n > n_0$ by specifying the inputs to the system for $n \geq n_0$.

Matrix state-difference equation

$$\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n)$$

Output equation

$$\mathbf{y}(n) = \mathbf{C}\mathbf{v}(n) + \mathbf{D}\mathbf{x}(n)$$

m-functions used

Function	**Purpose and Use**
`dlsim`	Given: state or transfer function (TF) model of discrete-time system, input, ICs, `dlsim` returns system output.
`dstep`	Given: state or TF model of discrete-time system, `dstep` returns step response.
`filter`	Given: DE or TF model of discrete-time system, input, ICs, `filter` returns system output.
`freqz`	Given: TF of discrete-time system, `freqz` returns frequency response.
`poly`	Given a square matrix, `poly` returns its characteristic equation. Given a vector of root values, `poly` returns a polynomial having these roots.
`residue`	Given: rational function $T(\sigma) = N(\sigma)/D(\sigma)$, `residue` returns roots of $D(\sigma) = 0$ and PF constants of $T(\sigma)$.
`residuez`	Given: rational function $T(z) = N(z)/D(z)$, `residuez` returns roots of $D(z) = 0$ and PF constants of $T(z)$.
`rlocus`	Given: an equation in the form $1 + K[\text{num}(\sigma)]/[\text{den}(\sigma)] = 0$, `rlocus` returns plot of locus of roots for K varying.
`roots`	Given: coefficients of polynomial p, `roots` returns roots of $p = 0$.

Problems

Reinforcement Problems

3.1. **Unit impulse response**. Find an analytical expression for the unit impulse response for each of the following systems.

(a) $y(n) - 0.2y(n-1) = x(n)$

(b) $y(n) - 0.2y(n-1) = 10x(n)$

(c) $y(n) + 0.2y(n-1) = x(n)$

(d) $y(n) + 0.2y(n-1) = 100x(n)$

(e) $y(n) - 1.0y(n-1) = x(n)$

(f) $y(n) - 1.1y(n-1) = x(n)$

(g) $y(n) - 1.1y(n-1) = x(n) + x(n-1)$

(h) $y(n) + 1.1y(n-1) = 17x(n)$

(i) $y(n) = 3x(n) - 2x(n-4) + x(n-5)$

(j) $y(n) = x(n) - 10x(n-1) + 8x(n-2) - 6x(n-4) + 4x(n-6)$

3.2. **System stability from the unit impulse response**. For each of the systems described in Problem 3.1, indicate whether the system is stable and explain your reasoning.

3.3. **Computer or state models**. For each of the systems characterized by the given difference equations, find a state-variable (first-order) representation. Include an output equation.

(a) $y(n) - y(n-2) = x(n)$

(b) $y(n) + y(n-2) = x(n)$

(c) $y(n) - 0.64y(n-1) - 0.7y(n-2) = x(n)$

(d) $y(n) + 0.65y(n-1) - 0.35y(n-2) - 0.11y(n-3) = x(n)$

3.4. **Stability tests**. An LTI system is modeled by the DE $y(n) + 0.5y(n-2) = x(n)$.

(a) Let $x(n) = 0$, $y(-2) = 1$, $y(-1) = 1$ and find $y(n)$ for $n = 0, 1, \ldots, 6$.

(b) Repeat part a for $x(n) = 0$, $y(-2) = -1$, $y(-1) = 2$.

(c) From the results of parts a and b, do you think the system is stable? If so, are the results sufficient to ensure stability? Explain.

(d) Now let $y(-2) = 0$, $y(-1) = 0$, and $x(n) = \delta(n)$. Find $y(n)$ for $n = 0, 1, \ldots, 6$.

(e) Repeat part d for $x(n) = u(n)$.

(f) Repeat part d for $x(n) = nu(n)$.

(g) From the results of parts d through f, do you think that the system is stable or unstable?

(h) Find the roots of the system's characteristic equation. Is the system stable? If so, how do you explain the behavior of the system in part f?

3.5. **Stability and unit impulse response**. Consider the system transfer function

$$H(z) = z^{-1}/(1 - 2z^{-1}), \quad |z| < 2.$$

(a) Is the system stable?

(b) Find $h(n)$

3.6. **Convolution by z-transforms**. An LTI system that has the unit impulse response $h(n) = (-0.80)^n u(n)$ is subjected to the input

$$x(n) = 2^n [u(n) - u(n-3)].$$

(a) Use z-transforms to find an analytical expression for the system output $y(n)$ due to the input $x(n)$.

(b) Use the m-function `conv` to obtain a numerical solution for $y(n)$.

(c) Compare your results from parts a and b for $n = 0, 5, 10$.

3.7. **Transfer function**. A linear time-invariant digital filter is described by the difference equation

$$y(n) - \sum_{k=1}^{3} \frac{1}{k} y(n-k) = \sum_{k=0}^{2} k x(n-k)$$

where k indicates the time delays of the filter.

(a) Find the filter transfer function $H(z) = Y(z)/X(z)$.

(b) What are the poles and zeros?

(c) Is the system stable?

3.8. **Causal and anticausal systems**. Suppose that a causal system has the unit impulse response $h_c(n) = r^n u(n)$ and that an anticausal system is described by $h_{ac}(n) = q^n u(-n-1)$, with $|r| < |q|$.

(a) If these two systems are connected together to form a new system, as shown in Figure 3-19, determine the transfer function $H(z)$ of this new system and give its region of convergence.

(b) What is the unit impulse response of the new system?

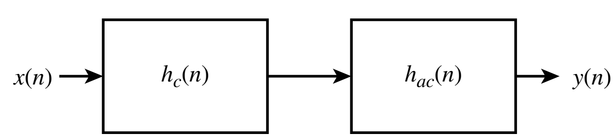


Figure 3-19: Cascade connection of a causal and an anticausal system

3.9. Impulse and step response using an m-function. A causal filter is described by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5z^2 - 0.9167z - 0.5417}{z^3 - 0.5833z^2 - 0.0417z + 0.0417}, \quad |z| > 0.50.$$

(a) Find the zeros and poles. Is it stable?

(b) Find an analytical expression for the unit impulse response of this system. Use `residue` or `residuez` for the PFE. Use this result to show whether or not this filter is stable.

(c) Determine an equation for the unit step response of this filter. Verify the steady-state output with the final value theorem. See Table 2.3.

(d) Use `filter` or an equivalent function to plot the unit impulse and unit step responses.

3.10. Cascade connection of two systems. Shown in Figure 3-20 is a cascade connection of two causal, linear systems. The input to the first system is $x(n)$ and its output is $w(n)$, the input to the second system is $w(n)$, and its output is $y(n)$. System 1 is modeled by $h_1(n) = a^n u(n)$ and $h_2(n) = b^n u(n)$. Assume that a and b satisfy the condition $0 < a < b < 1$.

(a) Find the transfer function $H(z) = Y(z)/X(z)$ and determine its poles and zeros. Is the overall system stable?

(b) For $x(n) = A\cos(n\pi/2)u(n)$, find $y_{ss}(n)$.

(c) Determine an analytical expression for the unit impulse response $h(n)$ of the overall system. Use an m-function to plot $h(n)$ for $a = 0.80$ and $b = 0.90$.

(d) Find an analytical expression for the unit step response $y(n)$ of the overall system. Plot $y(n)$ using the values of a and b in part c.

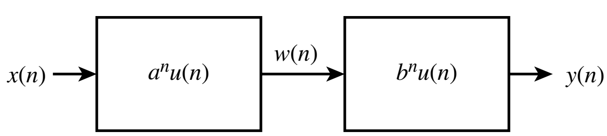


Figure 3-20: Two causal systems in a cascade connection

3.11. Convolution and region of convergence. Use z-transforms to evaluate the following convolutions. The results should be the same as obtained in Problem 2.9.

(a) $c_1(n) = u(n) * u(n)$

(b) $c_2(n) = u(-n-1) * u(-n-1)$

(c) $c_3(n) = u(-n-1) * r^n u(n), 0 < r < 1$

(d) $c_4(n) = u(n) * r^n u(n), 0 < r < 1$

3.12. Diagrams for moving-average filters. In Chapter 2, we used weighted moving-average filters described by

Weighted MA filter

$$y(n) = \frac{G}{L}\sum_{k=0}^{L-1} w(k)x(n-k)$$

with different weighting sequences, $w(k)$. Draw a nonrecursive diagram (SFG) to realize this filter for the following weighting sequences with $0 \le k \le L-1$. Use $L = 20$, and assign the gain factor G/L to an input or an output branch. See the **WHAT IF**? section following Example 2.2 to review the calculation of G/L.

(a) $w(k) = 1$

(b) $w(k) = 1 - 0.05k$

(c) $w(k) = (0.86)^k$

3.13. Transfer functions for moving-average filters. For each of the three systems in Problem 3.12, find the transfer function $H(z) = Y(z)/X(z)$.

3.14. Response of a second-order system. A causal LTI system is described by

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1),$$

where a_1, a_2, b_0, and b_1 are real constants.

(a) Find the values of a_1, a_2, b_0, and b_1 such that the zero-input response will be of the form
$y_{ZI}(n) = A\cos(n\pi/3 + \beta)$.

(b) Repeat part a for $y_{ZI}(n) = A\cos(2n\pi/3 + \beta)$.

(c) Repeat part a for $y_{ZI}(n) = A(0.5)^n + B(2)^n$.

(d) For $a_1 = -1$, $a_2 = 0.5$, and $b_0 = b_1 = 0.5$, find an analytical expression for $y(n)$ if $y(-1) = y(-2) = 0$ and $x(n) = 2\cos(n\pi/3)u(n)$.

3.15. Sinusoidal steady state. For the system of Problem 3.14,

(a) Find $H(z)$ in terms of a_1, a_2, b_0, and b_1.

(b) Use the sinusoidal steady-state formula to verify the steady-state part of the solution to Problem 3.14d.

3.16. Generic second-order filter. Second-order digital filters can be described by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - [2r\cos\beta]z^{-1} + r^2 z^{-2}}.$$

(a) Draw an SFG that describes this filter.

(b) What is the DE for this filter?

(c) Show that the filter poles are $z_{1,2} = re^{\pm j\alpha}$.

(d) Discuss stability from the point of view of r and β.

(e) Find an analytical expression for the unit impulse response $h(n)$.

(f) Suppose that the numerator of the transfer function is multiplied by z^{-2}. Show the change required in the system diagram and determine the new unit impulse response, denoted as $h'(n)$.

(g) Use an m-function to plot $h(n)$ and $h'(n)$ and comment on their similarities. (For the answer, we used $r = 0.9$ and $\beta = \pi/3$).

3.17. **System diagram and impulse response**. A digital lowpass filter is modeled by the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.067(z+1)^2}{z^2 - 1.145z + 0.414}.$$

(a) Draw a system diagram in the manner of Figure 3-10c.

(b) Find an analytical expression for the unit impulse response $h(n)$ of this filter.

(c) Verify a few values of your calculation by comparing them with $h(n)$, determined by the m-function `filter` or equivalent.

(d) Use Mason's Gain Rule to find $H(z) = Y(z)/X(z)$, which should yield the given transfer function.

3.18. **Transfer function, DE, zeros and poles, impulse response, and an m-function**. An SFG model for a Chebyshev digital filter is given in Figure 3-21.

(a) Use MGR to find the transfer function $H(z) = Y(z)/X(z)$.

(b) Find a difference equation for the system.

(c) Use `roots` to determine the filter's poles and zeros.

(d) Use an m-function to plot the unit impulse response $h(n)$. Would you estimate this to be a bandstop or a bandpass filter?

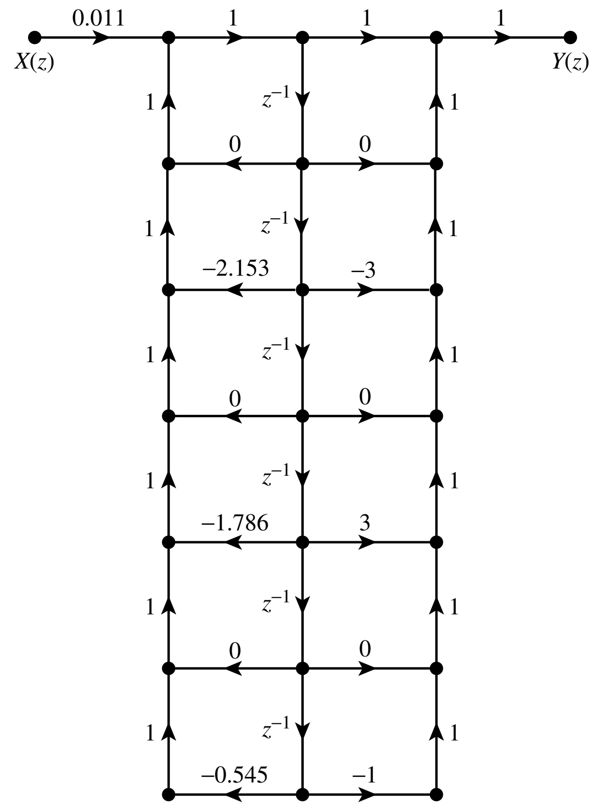


Figure 3-21: SFG Model for Chebyshev digital filter

Exploration Problems

3.19. **Poles and time response**. The output of an LTI system is affected by the input, the initial conditions, and the system characteristics. In terms of transforms we can describe the situation as $Y(z) = Y_{ZI}(z) + Y_{ZS}(z)$ or in the time domain, as $y(n) = y_{ZI}(n) + y_{ZS}(n)$. In this problem, let's look at the effect the poles of a transform have on the time-domain response. In any case, we have for a typical term in the partial fraction expansion

$$W(z) = \sum_{k=1}^{R} \frac{C_k z}{z - p_k} \quad \text{or} \quad w(n) = \sum_{k=1}^{R} C_k (p_k)^n u(n)$$

where $w(n)$ is the time response of interest (zero input, zero state, or combined). We are assuming distinct poles and an ROC that yields only causal terms. First, consider a term in $W(z)$ such as $Cz/(z-p)$. Give a description, and then, sketch the corresponding time-domain term for the following real values of p:

(a) $0 < p < 1$

(b) $p > 1$

(c) $-1 < p < 0$

(d) $p < -1$

(e) $p = 1$

(f) $p = -1$

Next, consider two complex poles such as $p = re^{\pm j\theta}$ and give a description and sketch the time-domain term for

(g) $r = 1$

(h) $r < 1$ and

(i) $r > 1$.

Include at least two different values of θ in part g.

3.20. **Poles and time response with an m-function**. We can use an m-function such as `filter` to revisit Problem 3.19 and explore such items as the magnitude of the pole p, the magnitude of the radius r, and the size of the angle θ. We simply consider typical partial fraction terms such as

$$W(z) = \frac{Cz}{z-p} \quad \text{or} \quad W(z) = \frac{Cz}{z-re^{j\theta}} + \frac{C^*z}{z-re^{-j\theta}}$$

as transfer functions and use a unit impulse input to observe the corresponding time response. For instance, for complex poles we can assume C to be real, which gives

$$W(z) = \frac{Cz}{z-re^{j\theta}} + \frac{C^*z}{z-re^{-j\theta}} = \frac{2Cz[z-r\cos(\theta)]}{z^2-2r\cos(\theta)z+r^2}$$

and the numerator and denominator polynomials are in the required form, for `residue`. Investigate parts a through i of Problem 3.19 by assuming $W(z) = H(z)$ and plotting $h(n)$ for different magnitudes of p, r, and θ.

Note: An alternative to writing an m-file is to use the Interactive Pole-Zero Placement visualization found at the website:

`http://www.ntspress.com/section.aspx?id=45&page=Linear%20Systems`

3.21. **Locus of the roots of an equation with an m-function**. In Problems 3.19 and 3.20, we noticed the effect of pole locations on the time response. In system design we often want to see the effect of varying a system's parameter, called G, on the values of the system's roots. Starting with the characteristic equation

$$\sum_{k=0}^{N} a_k z^k = 0$$

we collect all terms that involve the varying parameter G and form the polynomial $G \cdot \text{num}(z)$ and put the rest of the terms in the polynomial $\text{den}(z)$. Now the characteristic equation is in the form

$$G \cdot \text{num}(z) + \text{den}(z) = 0$$

or, dividing by $\text{den}(z)$

$$1 + \frac{G \cdot \text{num}(z)}{\text{den}(z)} = 0.$$

For instance, the CE $\alpha_3 z^3 + \alpha_2 z^2 + Gz^2 + \alpha_1 z + \alpha_0 + G = 0$ would be put in the form

$$G \cdot (z^2+1) + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$$

or

$$1 + \frac{G \cdot (z^2+1)}{\alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0} = 0,$$

where the $\alpha' s$ are known constants and, of course, some of them could be zero. A detailed discussion of this method, known as the root locus procedure, normally is part of an undergraduate control-systems course. We will not proceed further in the theory and ramifications of the root locus procedure except to point out that the m-function `rlocus` can be used to determine the effect of varying a system parameter on pole locations and hence on system response. Let's use this function to plot the locus of roots for some equations of interest to you. Some relatively simple ones for starters are.

(a) $z^2 + z + G = 0$

(b) $z^2 + Gz + 10 = 0$

(c) $z^3 + 3z^2 + 2z + G = 0$

In each case you should check some easy-to-find roots to see if the computer-generated plot is reasonable.

3.22. **Determining a system transfer function**. We want to design a causal LTI system that has the property that if the input is $x(n) = (1/2)^n u(n) - 0.25(1/2)^{n-1}u(n-1)$ then the output is $y(n) = (1/3)^n u(n)$. Find the system transfer function $H(z)$.

3.23. **An alternative inverse transform**. Up to this point, the function to be inverse transformed has been written in positive powers of z. In this problem, we consider an alternative using the transform $X(z)$ as a example where

$$X(z) = \frac{4z^2}{(z-0.5)(z+0.5)}, \quad |z| > |0.5|.$$

(a) Write this in terms of z^{-1}.

(b) Find the partial fraction expansion in the form

$$X(z) = \frac{C_1}{1-\alpha z^{-1}} + \frac{C_2}{1-\beta z^{-1}}$$

and evaluate the coefficients C_1 and C_2.

(c) Determine the sequence $x(n)$.

3.24. **A noncausal output**. The unit impulse response of a system is given by

$$h(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)$$

while the system is subjected to the input

$$x(n) = 4\delta(n) + 3\delta(n+1) + 2\delta(n+2) + 1\delta(n+3).$$

(a) Is the system stable? Explain.

(b) Is the system causal? Explain.

(c) Use z-transforms to find and plot the output sequence $y(n)$.

3.25. **Initial-value theorem**. For the z-transform $F(z)$ of a sequence that is zero for $n < 0$, we can write $F(z) = f(0) + f(1)z^{-1} + f(2)z^{-2} + \ldots$. Suppose you were given two polynomials in z and wanted to find $f(0)$ without doing long division to obtain $F(z)$ in the form just above or by taking the inverse transform of $F(z)$.

(a) Show that $f(0) = \lim_{z\to\infty} F(z)$.

(b) Use this result to find $f(0)$ for $F(z) = z/(z-1)$. Compare your result with the value of $f(0)$ obtained by long division and also by finding $f(n)$, which is the inverse transform of $F(z)$.

3.26. **Stability for a second-order system**. A causal, second-order system is described by the transfer function

$$H(z) = \frac{N(z)}{z^2 - a_1 z - a_2}.$$

In an $a_1 - a_2$ plane (as in Figure 3-22) show the region for a stable system.

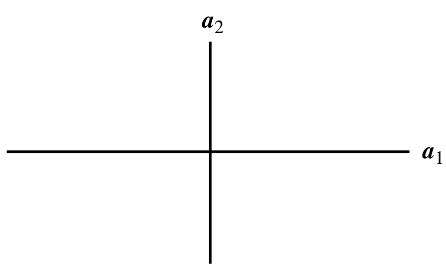


Figure 3-22: $\boldsymbol{a}_1 - \boldsymbol{a}_2$ plane

3.27. **Unit impulse response for an IIR system, N < L**. Starting from the transfer function,

$$H(z) = \sum_{k=0}^{L} b_k z^{-k} / \sum_{k=0}^{N} a_k z^{-k}$$

assume $a_0 = 1$ and distinct poles. Find an expression for the unit impulse response $h(n)$ for $N < L$.

Hint: Use the pattern in Section 3.2.2, but first obtain positive powers of z in $H(z)$ by multiplying both numerator and denominator of the right-side of the expression for $H(z)$ by z^L.

3.28. **A more general second-order filter**. Consider the filter described by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

where all coefficients are real.

(a) Draw an SFG for this filter.

(b) Use the SFG to write a set of state and output equations for the filter.

(c) Find the coefficients a_1 and a_2 for purely imaginary system eigenvalues of $\lambda_{1,2} = \pm j0.866$.

(d) We want to design a bandpass filter with its passband centered at $\theta = \pi/2$. The filter gain values at $\theta = 0$ and $\theta = \pi$ are to be zero. Use a_1 and a_2 from part c, and determine b_0, b_1, and b_2.

3.29. **A parallel system and potpourri**. A causal system is modeled by the signal flowgraph of Figure 3-23a.

(a) What is the order of the system?

(b) Write a set of state and output equations for this system.

(c) Find the characteristic equation for the system. (*Hint:* Use the m-function `poly`.)

(d) What are the characteristic roots? Is the system stable?

(e) The SFG for another implementation of this same system is shown in Figure 3-23b, with some unspecified gains. Determine the values of α, β, γ, and δ to make the two realizations have the same characteristic equation.

3.30. **State equations from system diagrams**.

(a) Write state and output equations in matrix form for the system characterized by the block diagram shown in Figure 3-24a.

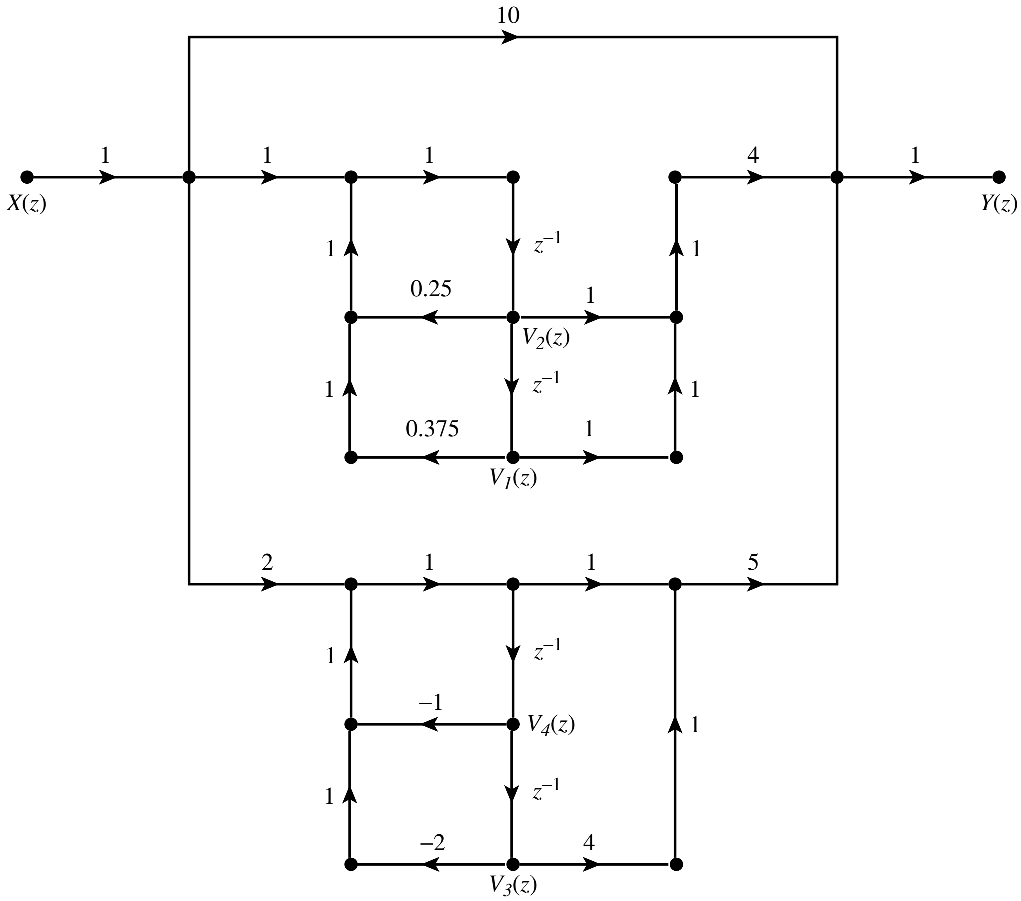
(b) Write state and output equations in matrix form for the system characterized by the signal flowgraph shown in Figure 3-24b. Notice that the system has two inputs.

(c) Write state and output equations in matrix form for the system characterized by the signal flowgraph shown in Figure 3-24c.

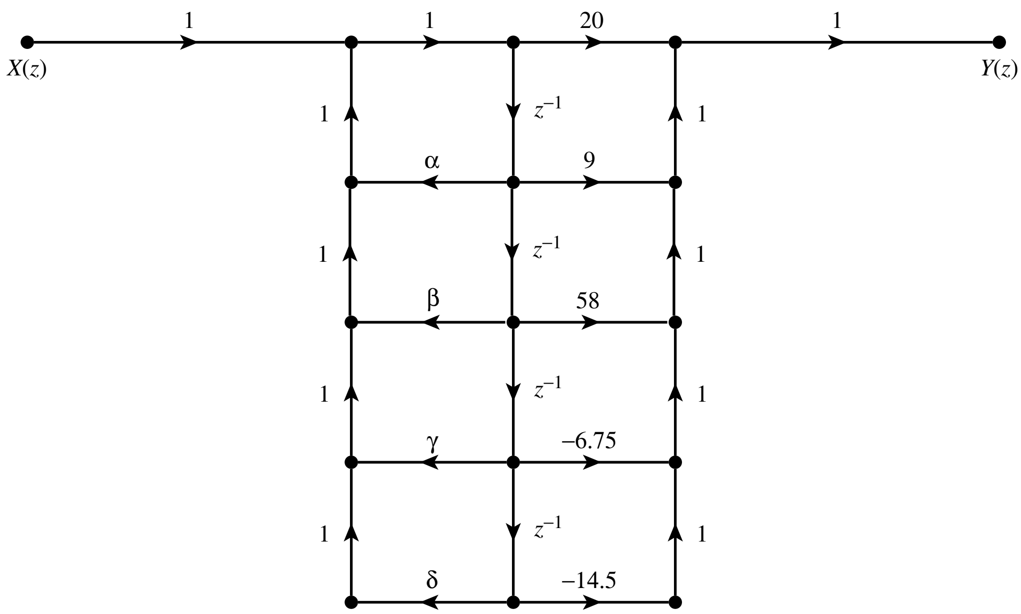
3.31. **Transfer functions from system diagrams**.

(a) Use Mason's Gain Rule to find the transfer function $H(z) = Y(z)/X(z)$ for the system whose signal flowgraph is shown in Figure 3-23a.

(b) Repeat part a for the system whose SFG is shown in Figure 3-23b. Your answer will be in terms of α, β, γ, and δ.



(a)



(b)

Figure 3-23: Signal flowgraphs for Problem 3.29

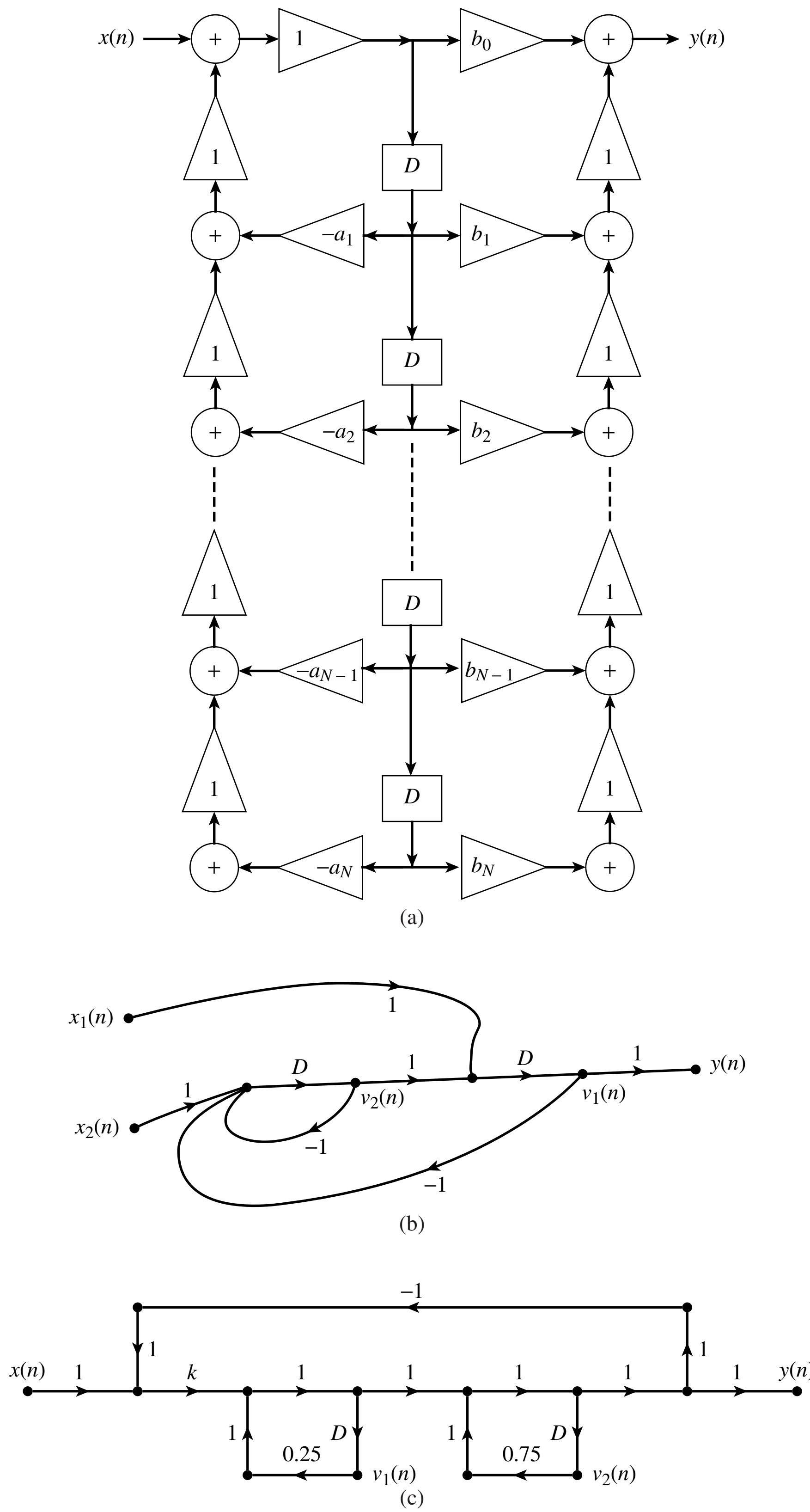


Figure 3-24: Signal flowgraphs for Problem 3.30

Annotated Bibliography

1. Doetsch, Gustav, *Guide To The Applications of The Laplace and z-Transforms*, Van Nostrand Reinhold Company, London, 1961, and 1971. *This is a lean but thorough treatise that was written early in the game* (1961) *to guide engineers through the pitfalls often encountered in applying Laplace transform methods to the solution of practical engineering problems. Professor Doetsch, University of Freiburg im Breisgau, introduces the z-transformation by using the traditional (mid-20th century) approach of defining* $z = e^{sT}$. *This book is full of historical footnotes and the author's "Rules" that pertain to the use of transform methods, Laplace and z.*

2. Gabel, Robert A., and Richard A. Roberts, *Signals and Linear Systems, 3rd ed.*, John Wiley & Sons, NY, 1987. *Chapter Four, "The z-Transform," covers all the usual theorems and properties and also gives a concise treatment of the inversion of z-transforms that includes a cleverly concealed use of the inversion integral. Applications to the steady-state analysis of linear systems and to the state-variable model are considered, and the chapter concludes with a short example that illustrates the use of the z-transform in the numerical solution of partial differential equations.*

3. O'Flynn, Michael, and Eugene Moriarty, *Linear Systems, Time Domain and Transform Analysis*, John Wiley & Sons, NY, 1987. *This interesting book contains material suitable for a first course in linear systems as well as for a more advanced course. Methods of modeling and analysis are developed for both continuous- and discrete-time systems. Chapter Seven includes a thorough treatment of bilateral z-transforms, including applications in linear systems with random and signal plus noise inputs, where correlation and spectral functions are considered.*

4. Oppenheim, Alan V., and Ronald W. Schafer, with John R. Buck, *Discrete-Time Signal Processing, Second Edition*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1999. *In Chapter Four of this popular digital signal processing text, the z-transform is developed as a generalization of the discrete-time Fourier transform. Evaluating the inverse transform by using contour integration is emphasized including applications to complex convolution and Parseval's theorem.*

5. Strum, Robert D., and Donald E. Kirk, *First Principles of Discrete Systems and Digital Signal Processing*, Addison-Wesley Publishing Company, Reading, MA, 1988. *Chapter Six develops the theory and applications of z-transforms. Included are detailed solutions of many examples.*

6. *Historical Reference* Gardner, Murray F., and John L. Barnes, *Transients in Linear Systems, Studied by the Laplace Transformation*, John Wiley & Sons, NY, 1942. *See Chapter Nine for a historical note about a method for solving difference equations that utilizes Laplace transforms rather than z-transforms.*

Answers

3.1.

(a) $h(n) = 0.2^n u(n)$

(b) $h(n) = 10 \cdot 0.2^n u(n)$

(c) $h(n) = (-0.2)^n u(n)$

(d) $h(n) = 100(-0.2)^n u(n)$

(e) $h(n) = 1^n u(n) = u(n)$

(f) $h(n) = (1.1)^n u(n)$

(g) $h(n) = (1.1)^n u(n) + (1.1)^{n-1} u(n-1)$

(h) $h(n) = 17(-1.1)^n u(n)$

(i) $h(n) = 3\delta(n) - 2\delta(n-4) + \delta(n-5)$

(j) $h(n) = \delta(n) - 10\delta(n-1) + 8\delta(n-2) - 6\delta(n-4) + 4\delta(n-6)$

3.2. All answers are based on the stability criterion of $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. Using this or any other criterion, parts e, f, g, and h are unstable systems; the others are stable.

3.3. For parts a, b, and c we define $v_1(n) = y(n-2)$, $v_2(n) = v_1(n+1)$, and $y(n) = v_2(n+1)$.

(a) $\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(n)$
$y(n) = [1 \;\; 0]\mathbf{v}(n) + x(n)$

(b) $\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(n),$
$y(n) = \begin{bmatrix} -1 & 0 \end{bmatrix} \mathbf{v}(n) + x(n)$

(c) $\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 \\ 0.7 & 0.64 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(n),$
$y(n) = \begin{bmatrix} 0.7 & 0.64 \end{bmatrix} \mathbf{v}(n) + x(n)$

(d) $\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.11 & 0.35 & -0.65 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x(n)$
$y(n) = \begin{bmatrix} 0.11 & 0.35 & -0.65 \end{bmatrix} \mathbf{v}(n) + x(n)$

3.4.

(a) $y(n) = \{-0.5000 \;\; -0.5000 \;\; 0.2500 \;\; 0.2500 \; \ldots -0.1250 \;\; -0.1250 \;\; 0.0625\}$

(b) $y(n) = \{0.5000 \;\; -1.0000 \;\; -0.2500 \;\; 0.5000 \;\; 0.1250 \;\; -0.2500 \;\; -0.0625\}$

(c) Seems to be stable; the zero-input responses $\to 0$ as n increases.

(d) $y(n) = h(n) = \{1.0000 \;\; 0 \;\; -0.5000 \;\; 0 \;\; 0.2500 \;\; 0 \;\; -0.1250\}$

(e) $y(n) = \{1.0000 \;\; 1.0000 \;\; 0.5000 \;\; 0.5000 \;\; 0.7500 \;\; 0.7500 \;\; 0.6250\}$

(f) $y(n) = \{0 \;\; 1.0000 \;\; 2.0000 \;\; 2.5000 \;\; 3.0000 \;\; 3.7500 \;\; 4.5000\}$

(g) Seems to be stable; $h(n)$ becomes smaller as n increases.

(h) $z_{1,2} = \pm j0.707$, stable, response is being driven toward ∞ by an unbounded input. The FPALS file `A3_4` gives solutions to the corresponding parts.

3.5.

(a) Yes, unit circle is included in the ROC.

(b) $h(n) = -0.5\delta(n) - 0.5(2)^n u(-n-1)$

3.6.

(a) $y(n) = -3.75\delta(n) + 5\delta(n-1) + 4.75(-0.8)^n u(n)$

(b) See FPALS file `A3_6`.

(c) From part a, $y(0) = 1$, $y(5) = -1.557$, $y(10) = 0.5100$; the results from file `A3_6` are identical.

3.7.

(a) $H(z) = (z^2 + 2z)/(z^3 - z^2 - 0.500z - 0.333)$

(b) Poles: $z_1 = 1.49$, $z_{2,3} = 0.473e^{\pm j2.111}$, zeros $z_1 = 0$, $z_2 = -2$

(c) Causal system, unstable

3.8.

(a) $H(z) = -z^2/[(z-r)(z-q)]$, ROC: $|r| < |z| < |q|$

(b) $h(n) = [1/(q-r)][r^{n+1}u(n) + q^{n+1}u(-n-1)]$

3.9.

(a) Poles: $z_1 = 0.500$, $z_2 = 0.334$, $z_3 = -0.250$; zeros: $z_1 = -0.250$, $z_2 = 0.433$; the system is stable because the unit circle is included in ROC.

(b) $h(n) = [4(0.50)^n - 9(0.33)^n]u(n) - 13\delta(n)$

(c) $y(n) = [8.5 - 4.5(0.334)^n - 4.0(0.5)^n]u(n)$
$y(\infty) = (z-1)Y(z)|_{z-1} = 8.50$ as in time domain

(d) See FPALS file `A3_9` for both plots.

3.10.

(a) $H(z) = H_1(z)H_2(z) = Y(z)/X(z) = z^2/[(z-a)(z-b)]$, 2 zeros at $z = 0$ and poles at $z_1 = a$ and $z_2 = b$; stable because both poles of this causal system are inside the unit circle

(b) $y_{ss}(n) = AM\,cos(n\pi/2 + P)$, where

$$M = 1/\sqrt{(1-ab)^2 + (a+b)^2}$$

and

$$P = \tan^{-1}[(a+b)/(ab-1)]$$

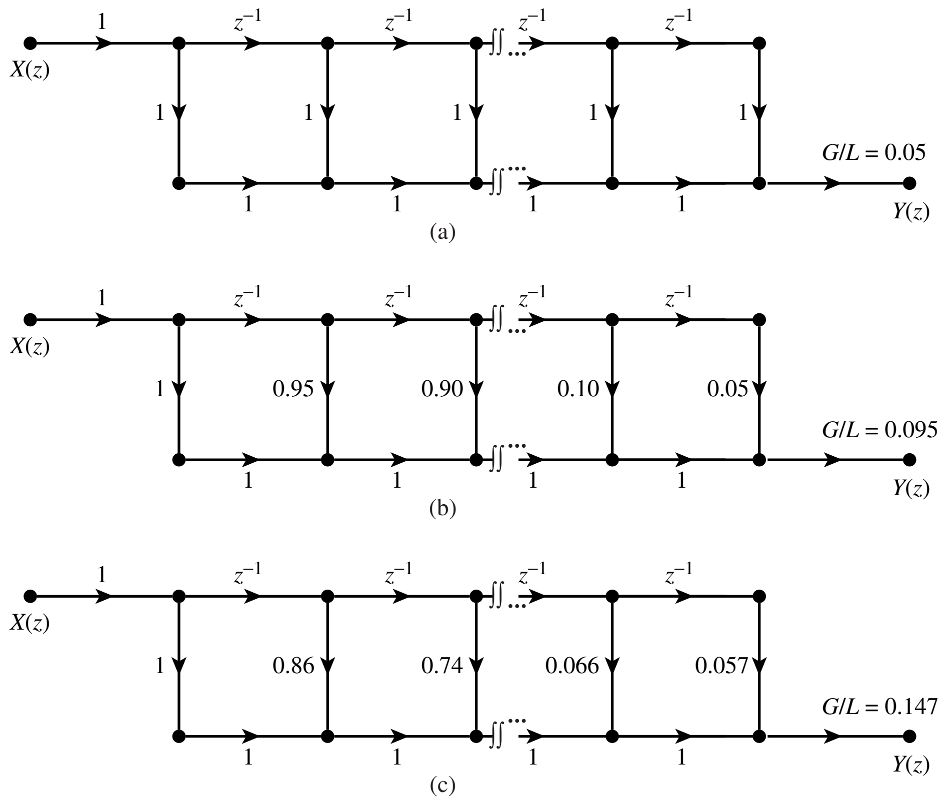


Figure 3-25

(c) $h(n) = [1/(b-a)] \cdot (b^{n+1} - a^{n+1})u(n)$; see FPALS file `A3_10`

(d) $y(n) = 1/[(1-a)(1-b)] + [a^2/[(a-b)(a-1)]] \cdot a^n + \ldots [b^2/[(b-a)(b-1)]] \cdot b^n$, $n \geq 0$; see FPALS file `A3_10`

3.11.

(a) $c_1(n) = [n+1] \cdot u(n)$

(b) $c_2(n) = [-n-1] \cdot u(-n-1)$

(c) $c_3(n) = [1/(1-a)] \cdot [u(-n-1) + a^{n+1}u(n)]$

(d) $c_4(n) = [1/(1-a)] \cdot [1 - a^{n+1}]u(n)$

3.12. See Figure 3-25.

3.13.

(a) $H(z) = 0.05[1 + z^{-1} + z^{-2} + \ldots + z^{-19}]$

(b) $H(z) = 0.095[1 + 0.95z^{-1} + 0.90z^{-2} + \ldots + 0.05z^{-19}]$

(c) $H(z) = 0.147[1 + 0.86z^{-1} + 0.74z^{-2} + \ldots + 0.057z^{-19}]$

3.14.

(a) b_0, b_1 any values, $a_1 = -1$, $a_2 = 1$

(b) b_0, b_1 any values, $a_1 = 1$, $a_2 = 1$

(c) b_0, b_1 any values, $a_1 = -2.5$, $a_2 = 1$

(d) $2.24(0.707)^n \cos(n\pi/4 + \pi/3) + 3.46 \cos(n\pi/3 - \pi/2)$, $n \geq 0$

3.15.

(a) $H(z) = Y(z)/X(z) = (b_0 + b_1 z^{-1})/(1 + a_1 z^{-1} + a_2 z^{-2})$

(b) $H(z) = (0.5 + 0.5z^{-1})/(1 - z^{-1} + 0.5z^{-2})$,
$H(e^{j\pi/3}) = 1.73e^{-j1.57}$,
$y_{ss}(n) = 3.46 \cos(n\pi/3 - 1.57)$

3.16.

(a) See Figure 3-26a.

(b) $y(n) - [2r \cos\beta]y(n-1) + r^2 y(n-2) = x(n)$

(c) $z_{1,2} = r\cos\beta \pm \sqrt{[r\cos\beta]^2 - r^2} = r(\cos\beta \pm j\sin\beta) = re^{\pm j\beta}$

(d) β may take on any value with $|r| < 1$

(e) $h(n) = [1/\sin\beta]r^n \sin(\beta[n+1])u(n)$

(f) See Figure 3-26b. $h'(n) = [1/\sin\beta]r^{n-2}\sin(\beta[n-1])\, u(n-2)$

(g) See FPALS file `A3_16`. $h'(n)$ is the same as $h(n)$, except delayed by two samples.

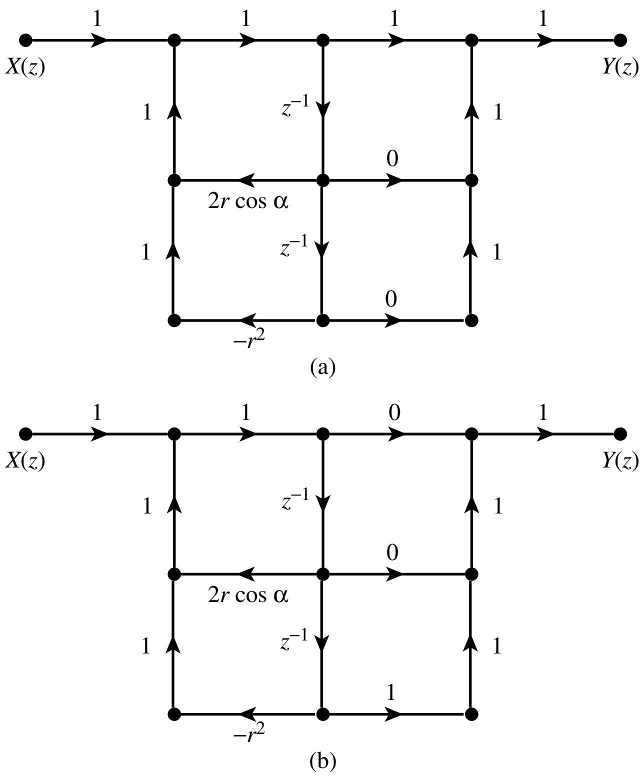


Figure 3-26

3.17.

(a) See Figure 3-27.

(b) $h(n) = 0.162\delta(n) + 0.908(0.643)^n \cos(0.474n - 1.675)$, $n \geq 0$ with $h(0) = 0.0675, h(1) = 0.211, \ldots$; run FPALS file `A3_17`.

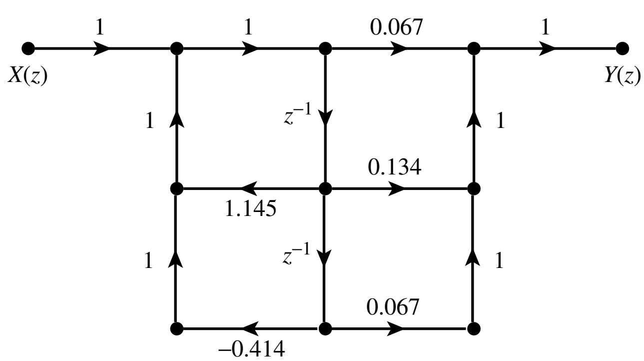


Figure 3-27

3.18.

(a) $Y(z)/X(z) = \frac{1+2.153z^{-2}+1.786z^{-4}+0.545z^{-6}}{0.011[1-3z^{-2}+3z^{-4}-z^{-6}]}$

(b) $y(n)+2.153y(n-2)+1.786y(n-4)+0.545y(n-6) = \ldots 0.011[x(n) - 3x(n-2) + 3x(n-4) - x(n-6)]$

(c) Zeros: $z_{1,2,3} = +1$, $z_{4,5,6} = -1$; poles: $z_{1,2} = 0.276 \pm j0.888$, $z_{3,4} = \pm j0.853$, $z_{5,6} = -0.276 \pm j0.888$

(d) See FPALS file `A3_18`; probably bandpass.

3.19. First

(a) $C(p)^n u(n)$, monotonically decreasing in amplitude

(b) $C(p)^n u(n)$, monotonically increasing in amplitude

(c) $C(p)^n u(n)$, alternating in sign but decreasing in amplitude

(d) $C(p)^n u(n)$, alternating in sign but increasing in amplitude

(e) $Cu(n)$, a step sequence

(f) $C(-1)^n u(n)$, an alternating step sequence

(g) $A\cos(\theta n + \alpha)u(n)$, undamped oscillation

(h) $Ar^n \cos(\theta n + \alpha)u(n)$, decreasing oscillation

(i) $Ar^n \cos(\theta n + \alpha)u(n)$, increasing oscillation

See the plots in the answers to 3.20.

3.20. We used `filter` to plot the unit impulse response.

(a) $h(n) = (0.9)^n u(n)$ and $h(n) = (0.5)^n u(n)$

(b) $h(n) = (1.1)^n u(n)$ and $h(n) = (1.2)^n u(n)$

(c) $h(n) = (-0.9)^n u(n)$ and $h(n) = (-0.5)^n u(n)$

(d) $h(n) = (-1.1)^n u(n)$ and $h(n) = (-1.2)^n u(n)$

(e) $h(n) = 1^n u(n) = u(n)$

(f) $h(n) = (-1)^n u(n)$

(g) $h(n) = 2\cos(n\pi/6 - 1.047)$ and $h(n) = 2(0.577)\cos(n\pi/3 - 0.525)$

(h) $h(n) = 2(0.9)^n \cos(n\pi/6 - 1.047)$

(i) $h(n) = 2(1.1)^n \cos(n\pi/6 - 1.047)$; see FPALS file `A3_20`.

3.21. See FPALS file `A3_21`.

3.22. $[z(z - 0.5)]/[(z - 0.33)(z - 0.25)]$.

3.23.

(a) $4/[(1 - 0.5z^{-1})(1 + 0.5z^{-1})]$

(b) $C_1 = 2, C_2 = 2$

(c) $[2(0.5)^n + 2(-0.5)^n]u(n)$

3.24.

(a) Yes, the unit impulse response is absolutely summable, i.e., $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. From the point of view of z-transforms there are four poles at $z = 0$, inside the unit circle.

(b) The system is causal, $h(n) = 0, n < 0$

(c)

$$
\begin{aligned}
H(z) &= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4},\\
X(z) &= z^3 + 2z^2 + 3z + 4,\\
Y(z) &= H(z)X(z)\\
&= z^3 + 3z^2 + 6z + 10 + 10z^{-1}\\
&\quad + 9z^{-2} + 7z^{-3} + 4z^{-4},\\
y(n) &= \delta(n+3) + 3\delta(n+2) + 6\delta(n+1) + 10\delta(n)\\
&\quad + 10\delta(n-1) + 9\delta(n-2)\\
&\quad + 7\delta(n-3) + 4\delta(n-4).
\end{aligned}
$$

3.25.

(a) $\lim_{z\to\infty} F(z) = f(0) + \lim_{z\to\infty}[f(1)z^{-1} + f(2)z^{-2} + \ldots]$. But the second term on the right is zero giving $f(0) = \lim_{z\to\infty} F(z)$.

(b) $\lim_{z\to\infty} \dfrac{z}{z-1} = \lim_{z\to\infty} \dfrac{1}{1-1/z} = 1$; from the inverse transform: $f(n) = 1^n u(n)$, $f(0) = 1$; from long division: $z/(z-1) = 1 + 1/(z-1)$, and $f(0) = 1$.

3.26. For complex roots, $a_1^2/4 + a_2 < 0$ and $z_{1,2} = a_1/2 \pm j\sqrt{(a_1^2/4) + a_2}$ and for stability, $-1 < a_2 < 1$ and $(-a_1^2/4) > a_2$ and for real roots, $a_2 \le -(1 - a_1)$ and $a_2 \le (1 + a_1)$.

3.27. $y(n) = h(n) = \sum_{k=0}^{L-N} A_k\delta(n-k) + \sum_{k=1}^{N} C_k(p_k)^n.$

3.28.

(a) See Figure 3-28.

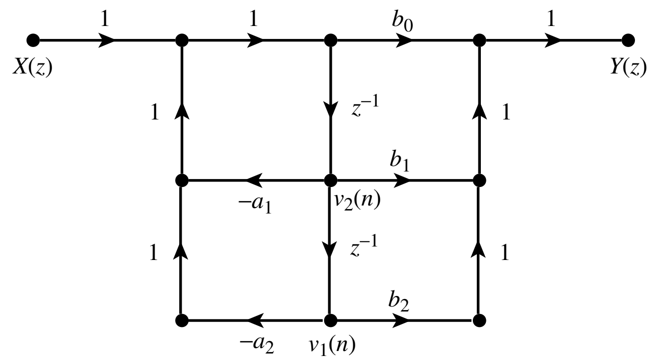


Figure 3-28

(b)

$$
\begin{aligned}
v_1(n+1) &= v_2(n),\\
v_2(n+1) &= -a_2v_1(n) - a_1v_2(n) + x(n),\\
y(n) &= [b_2 - b_0a_2]v_1(n)\\
&\quad + [b_1 - b_0a_1]v_2(n) + b_0x(n)
\end{aligned}
$$

(c) $a_1 = 0, a_2 = 0.75$

(d) $b_0 = -b_2$, so let $b_0 = 1$, $b_2 = -1$ and $b_1 = 0$

3.29.

(a) Fourth order.

(b) The states were assigned as the outputs of the delays from top to bottom in the SFG as $v_2(n)$, $v_1(n)$, $v_4(n)$, $v_3(n)$, respectively. $v_1(n+1) = v_2(n)$, $v_2(n+1) = 0.375v_1(n) + 0.25v_2(n) + x(n)$,

$$
\begin{aligned}
v_3(n+1) &= v_4(n),\ v_4(n+1)\\
&= -2v_3(n) - v_4(n) + 2x(n),\\
y(n) &= 4v_1(n) + 4v_2(n)\\
&\quad + 10v_3(n) - 5v_4(n) + 20x(n)
\end{aligned}
$$

(c) $z^4 + 0.75z^3 + 1.375z^2 - 0.875z - 0.75 = 0$

(d) $z_{1,2} = -0.500 \pm j1.323$, $z_3 = 0.750$, $z_4 = -0.500$, unstable causal system;

(e) $\alpha = -0.75$, $\beta = -1.375$, $\gamma = 0.875$, $\delta = 0.75$

3.30.

(a)

$$
\mathbf{v}(n+1) = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\\ -a_N & -a_{N-1} & -a_{N-2} & -a_{N-3} & \cdots & -a_1 \end{bmatrix}\mathbf{v}(n) + \begin{bmatrix}0\\0\\0\\0\\ \vdots\\1\end{bmatrix}x(n)
$$

$$
\begin{aligned}
y(n) = \big[&(b_N - b_0a_N) \quad (b_{N-1} - b_0a_{N-1}) \cdots\\
&(b_2 - b_0a_2) \quad (b_1 - b_0a_1)\big]\,\mathbf{v}(n) + b_0x(n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\begin{bmatrix} v_1(n+1)\\ v_2(n+1)\end{bmatrix} &= \mathbf{v}(n+1)\\
&= \begin{bmatrix} 0 & 1\\ -1 & 1\end{bmatrix}\mathbf{v}(n) + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(n)\\ x_2(n)\end{bmatrix}\\
y(n) &= \begin{bmatrix}1 & 0\end{bmatrix}\mathbf{v}(n) + 0\cdot\mathbf{x}(n)
\end{aligned}
$$

(c)

$$\begin{aligned}
\begin{bmatrix} v_1(n+1) \\ v_2(n+1) \end{bmatrix} &= \mathbf{v}(n+1) \\
&= \begin{bmatrix} \dfrac{0.25}{k+1} & \dfrac{-0.75k}{k+1} \\ \dfrac{0.25}{k+1} & \dfrac{0.75}{k+1} \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} \dfrac{k}{k+1} \\ \dfrac{k}{k+1} \end{bmatrix} x(n) \\
y(n) &= \begin{bmatrix} \dfrac{0.25}{k+1} & \dfrac{0.75}{k+1} \end{bmatrix} \mathbf{v}(n) + \frac{k}{k+1} x(n)
\end{aligned}$$

3.31. **(a)**

$$\begin{aligned}
H(z) &= \frac{20 + 9z^{-1} + 58z^{-2} - 6.75z^{-3} - 14.5z^{-4}}{1 + 0.75z^{-1} + 1.375z^{-2} - 0.875z^{-3} - 0.75z^{-4}} \\
&= \frac{20z^4 + 9z^3 + 58z^2 - 6.75z - 14.5}{z^4 + 0.75z^3 + 1.375z^2 - 0.875z - 0.75}
\end{aligned}$$

(b)

$$\begin{aligned}
H(z) &= \frac{20 + 9z^{-1} + 58z^{-2} - 6.75z^{-3} - 14.5z^{-4}}{1 - \alpha z^{-1} - \beta z^{-2} - \gamma z^{-3} - \delta z^{-4}} \\
&= \frac{20z^4 + 9z^3 + 58z^2 - 6.75z - 14.5}{z^4 - \alpha z^3 - \beta z^2 - \gamma z - \delta}
\end{aligned}$$